Ast/Phys 5022
Fall 2008
Problem Set #1 (due Tue Sep 16)

1. The divergence of fluid stress-energy tensor is zero in the absence of external forces:

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = \sum_{\beta=0,3} \frac{\partial}{\partial x^\beta} T^{\alpha\beta} = 0.$$

(In the first expression the Einstein summation convention, over the repeated index, β, is used).

In class we showed that when $\alpha = 0$, the above equation becomes the statement of the conservation of mass, or the continuity equation:

$$\frac{\partial}{\partial x^\beta} T^{0\beta} = \gamma^2 c^2 \frac{\partial}{\partial x^\beta}(\rho, \rho[\vec{v}/c]) = \gamma^2 c^2 \Big(\frac{\partial \rho}{\partial [ct]} + \frac{\partial [\rho v_x/c]}{\partial x} + \frac{\partial [\rho v_y/c]}{\partial y} + \frac{\partial [\rho v_z/c]}{\partial z} \Big)$$

$$= \gamma^2 c \Big(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot [\rho \vec{v}] \Big) = 0$$

Show that when $\alpha = 1, 2, 3$, the equation gives the 3 components of the equation of motion, i.e. the conservation of linear momentum.

2. In this problem we will demonstrate that there is no difference between covariant and contravariant tensor transformation in Eucledean space. Let's work in 2D space, and consider rotation in the anti-clockwise direction by angle θ that takes you from the unprimed to the primed coordinate system:

$$\begin{pmatrix} dx' \\ dy' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

A tensor transformation as it applies to this contravariant tensor, reads:

$$dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy \qquad \text{and} \qquad dy' = \frac{\partial y'}{\partial x} dx + \frac{\partial y'}{\partial y} dy.$$

A tensor transformation as it applies to a gradient of a scalar function, a covariant tensor, reads:

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} \qquad \text{and} \qquad \frac{\partial f}{\partial y'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'}.$$

Write down the matrices of transformation coefficients for (dx, dy) and $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, respectively, and show that they are the same.

3. In this problem we will demonstrate that there is a difference between covariant and contravariant tensor transformation in Minkowskian space. Let's work in 2D space, and consider a transformation from frame S to frame S', where S' is moving with respect to $+x$-axis with velocity v, and $\beta = v/c$.

$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}$$

Write down the matrices of transformation coefficients for (cdt, dx) and $(\frac{\partial f}{c\partial t}, \frac{\partial f}{\partial x})$, respectively, and show that they are NOT the same.