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`Note: Sections 7 & 8 were not covered in class and will not be on the final`

Part I

Structure formation

1 Introduction

1.1 Seeds for structure formation

If the Universe were perfectly smooth at all times on all scales, then the existence of the present day inhomogeneities (galaxies, clusters,....) would be very hard to explain. So we need some mechanism to produce seed fluctuations; these can later grow, due to gravity, into what we see today: a clustered distribution of galaxies and clusters of galaxies.

There are two general ways to seed the structure: inflation and topological defects left behind by early phase transitions. However, the later are disfavored by the data, for example, CMB power spectrum is not consistent with topological defects. We will consider only inflation-induced perturbations. During inflation quantum fluctuations got exponentially stretched to classical length scales. The power spectrum of fluctuations has Harrison-Zel'dovich shape (scale-invariant) if Hubble parameter was more or less constant during inflation. Inflation took place in the very early Universe, before recombination and before the present large scale structure emerged, so its signatures should be observable in the data such as CMB and matter power spectra.

1.2 Why 'expanding background' is important

Imagine a large *static* collection of mass particles. Whatever fluctuations in the density happen to be present in the beginning will get amplified by gravity. At some point in time all the particles will belong to different 'clouds', and each cloud will collapse further. At the same time clouds will collect into larger and larger clouds until there is one large object left.

The moral of the story is that if gravity is the only force on large scales, it always wins at the end, resulting in one large dense collection of matter. Fortunately, our Universe is different. The crucial difference is that the Universe is expanding and thus provides something for gravity to fight against. As a positive density bump tries to collapse due to its self-gravity the expansion of the Universe tries to pull it apart. This makes the growth rate of matter clumps in our Universe slower than in one with no expansion. The Jeans gravitational instability analysis shows that in the absence of expansion the growth of small density perturbations is exponential in time, while in the presence of Hubble expansion it is a power law in time.

In this section we will study what gravity can do given a Hubble time.

1.3 The plan for this set of lecture notes

Our goal here is to form dark matter halos and describe their properties.

All dark matter halos have baryons in them, which will form luminous stars, during or after halo collapse. Depending on the mass of the halo, we call these dark matter halos galaxies ($10^8 \lesssim M \lesssim 10^{12} M_\odot$), or clusters of galaxies ($10^{12} \lesssim M \lesssim 10^{14} M_\odot$). Groups of galaxies have masses at the low end of cluster masses. We will not discuss star formation, so for us galaxies and clusters are the corresponding dark matter halos.

After describing the Jeans gravitational instability in §2 and the linear growth of perturbations in §3 we will talk about the properties of dark matter halos: (i) halos density structure in the quasi-linear regime ($1 \lesssim \delta \lesssim 10-100$), §4.1; (ii) density structure in the non-linear regime ($\delta \lesssim 200$) which is usually addressed with the help of computer simulations and will not be discussed here;

(iii) DM halo mass function $n(M).dM$, in §5; (iv) spatial distribution and clustering of halos, §6. For the purposes of (iii) and (iv) dark matter halos can be thought of as structureless point masses.

2 Jeans analysis of gravitational instability and its applications

2.1 Jeans instability: linear perturbation analysis

All potential (and density) fluctuations started out smallm as we know from CMB. Present day density contrast within central parts of galaxies are $\sim 10^5$, and about 10^3 in clusters, so we have to account for many orders of magnitude of growth in fractional overdensity. We will start with linear analysis which will take us from very small $\delta\rho/\rho$ to about 1. For $\delta\rho/\rho > 1$ the evolution is no longer linear, which means that different k-modes do not evolve independently. Jeans analysis deals with the growth of the amplitude of density fluctuations in the linear regime so long as $\delta\rho/\rho \lesssim 1$; in section 4.1 we will discuss a simplified version of quasi-linear evolution.

Let us take a small amplitude perturbation in the density field, i.e. $\delta = \delta\rho/\rho \ll 1$. We start with the continuity, Euler's and Poisson equations in physical/proper coordinates, $\vec{r}$. Our treatment will be non-relativistic, i.e. we deal with scales smaller than Hubble length. We allow for the possibility of fluid pressure p.

$$(\frac{\partial\rho}{\partial t})_{\vec{r}} + \vec{\nabla}_{\vec{r}} \cdot (\rho\vec{u}) = 0, \tag{1}$$

$$\frac{d\vec{u}}{dt} = (\frac{\partial\vec{u}}{\partial t})_{\vec{r}} + (\vec{u}\cdot\vec{\nabla}_{\vec{r}})\vec{u} = -\vec{\nabla}_{\vec{r}}\Phi - \frac{1}{\rho}\vec{\nabla}_{\vec{r}}\, p \tag{2}$$

$$\nabla^2_{\vec{r}}\Phi = 4\pi G\rho \tag{3}$$

Next, we introduce a perturbation: a small density bump,

$$\rho(\vec{r},t) = \rho_0(t)[1+\delta(\vec{r},t)]; \tag{4}$$

subscript 0 means average background density at epoch t. Because of the density bump the material in the vicinity will then have associated peculiar velocities, i.e. velocities *not* due to the pure Hubble flow. The total velocity of a particle is

$$\vec{u} = d(a\vec{x})/dt = \dot{a}\vec{x} + \vec{v}(\vec{x},t), \tag{5}$$

where $v = a\dot{\vec{x}}$ is the peculiar velocity; $\dot{a}\vec{x}$ is the Hubble flow velocity. There will also be an associated peculiar gravitational potential, ϕ, which is different from that of a smooth unperturbed region:

$$\Phi = \Phi_0 + \phi \tag{6}$$

These perturbed quantities can be used instead of the corresponding "smooth" unperturbed quantities ρ_0, $\vec{u}$, Φ_0 in eqs. 1, 2, and 3. First, we rewrite the pressure term using the definition of acoustic velocity:

$$\frac{1}{\rho}\vec{\nabla}_{\vec{r}}p = \frac{1}{a\rho_0(1+\delta)}\frac{\partial p}{\partial\rho}\vec{\nabla}\rho_0(1+\delta) = \frac{1}{a(1+\delta)}\, c_s^2\vec{\nabla}\delta \tag{7}$$

We now convert the perturbed equations from physical/proper to comoving/coordinate coordinates in order to 'take out' the effects of the Hubble expansion. Comoving coordinates are related to proper coords by: $\vec{x} = \vec{r}/a(t)$, and spatial gradients are related by $\nabla_{\vec{x}} = a\nabla_{\vec{r}}$ (to simplify notation we will omit subscript $\vec{x}$ from now on). To convert time derivatives of an arbitrary function f at a fixed $\vec{r}$ to that at a fixed $\vec{x}$ we have to take Hubble flow into account:

$$(\frac{\partial f}{\partial t})_{\vec{r}} + H\vec{r}\cdot\vec{\nabla}_{\vec{r}}f = (\frac{\partial f}{\partial t})_{\vec{x}} \tag{8}$$

Next, we simplify the equations; to get the equation of motion and Poisson equations we subtract the unperturbed equations from the perturbed ones:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla} \cdot [(1+\delta)\vec{v}] = 0 \tag{9}$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{a}\vec{\nabla}\phi - \frac{1}{a(1+\delta)}\, c_s^2 \vec{\nabla}\delta \tag{10}$$

$$\nabla^2 \phi = 4\pi G \rho_0 a^2 \delta \tag{11}$$

Now we linearize continuity and Euler's equations, i.e. delete the terms that are smaller than small perturbed quantities, δ, ϕ, $\vec{v}$:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot \vec{v} = 0; \qquad \frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}\nabla\phi + \frac{1}{a}\, c_s^2 \vec{\nabla}\delta = 0 \tag{12}$$

Peculiar velocity, $\vec{v}$ can be eliminated between the above two equations by subtracting $\frac{1}{a}$ times the divergence of the second from the time derivative of the first, and then substituting linearized Poisson equation (dots are derivatives with respect to time):

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G \rho_0 \delta - \frac{c_s^2}{a^2}\nabla^2 \delta = 0 \tag{13}$$

As long as δ is small, the spatially varying δ field can be decomposed into Fourier modes, and each mode's time evolution can be considered separately. Each mode is characterized by a wavenumber k, or wavelength λ (such that $k\lambda = 2\pi$), and each has its own amplitude, δ_k. Eq. 13 can be rewritten for a single Fourier mode. The $\nabla^2\delta$ term is simplified by the fact that in Fourier space differentiation is equivalent to multiplication by ik,

$$\delta(\vec{x}) = \sum \delta_k e^{-ik\cdot\vec{x}} \quad \rightarrow \quad \vec{\nabla}\delta(\vec{x}) = \sum -ik\ \delta_k\ e^{-ik\cdot\vec{x}} \tag{14}$$

and so ∇^2 is equivalent to multiplication by $-k^2$. For a single mode, $\nabla^2\delta_k = -k^2\delta_k$:

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \Big[\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_0\Big]\delta_k = 0. \tag{15}$$

This is the linearized time evolution equation for density perturbations, δ_k, and applies to any type of cosmological model. Two types of solutions are possible, depending on which of the two terms in the square brackets dominates. If the gravitational potential term dominates then eq. 15 (without the gas pressure term) has two solutions, corresponding to the growing and decaying modes. (Note: the use of the word mode here is not to be confused with the k-mode usage!). Growing mode will eventually always win even if the decaying mode started out having the larger amplitude. For an arbitrary cosmology, i.e. an arbitrary set of Ω's the solution to the above equation is best obtained numerically. For E-dS and other simple models results can be obtained analytically. Let's look at some applications of eq. 15

2.2 Applications of the Jeans analysis

2.2.1 Jeans length

Equation 15 can have two different types of solutions depending on the sign of the last term, the one in square brackets.

• If the sign is positive, then solutions for δ_k are of sinusoidal type, which says that at any given place, or point in time, the density excess, δ_k oscillates. The sign of the square-brackets term is positive when the pressure term, $c_s^2 k^2/a^2$, dominates. The physical interpretation is that the pressure forces are strong enough to effectively resist gravity.Hence the oscillations.

• If the sign of the term is negative, then δ_k will have solutions that are not oscillating, but monotonic in time, for example solutions like hyperbolic sine or cosine are possible. These non-oscillatory solutions lead to a monotonic increase or decrease in the density excess. The physical interpretation is that gravity takes over and collapses the density perturbation before pressure forces can dissipate it. Thus, monotonic increase in δ_k can result in a bound structure at some later time.

The dividing line between these two types of behavior is the Jeans scale, when the square-brackets terms is zero: $k_J = \sqrt{4\pi G \rho_0}\, a/c_s$. The corresponding Jeans length is $\lambda_J = 2\pi/k_J$, and Jeans mass is $M_J = \rho_0 \lambda_J^3$.

2.2.2 Growth of DM perturbations in an Einstein-de Sitter Universe

In the next few sections we will be considering growth of density fluctuations in the dark matter alone. Because dark matter does not have any pressure of its own, and it does not couple to photons, the pressure term in the Jeans equation, $c_s^2 k^2/a^2$, will be ignored.

As usual, if we assume E-dS model, i.e. $\Omega_{matter} = 1$ things simplify a lot. Using $\rho_0 = \rho_{crit} = 3H_0^2/8\pi G$ and $t = 2/3H$, eq. 15 and its exact solution become

$$\frac{\partial^2 \delta_k}{\partial t^2} + \frac{4}{3t}\frac{\partial \delta_k}{\partial t} = \frac{2}{2t^2}\delta_k; \quad \delta_k = At^{2/3} + Bt^{-1}, \tag{16}$$

where A and B are constants. For the growing mode (the first of two terms in the solution) $\delta_k \propto a$, that is, the amplitude of fluctuations grows proportionately to the scale factor of the universe. This is an important result. Notice that the growth rate is independent of the value of k. This means that the spatial regions of the Universe whose $\delta(\vec{x})$ is composed of a range of k-modes, each with its own amplitude δ_k also grow at the rate $\propto a$.[1]

The growth in the linear regime (in the spatial regions where $\delta \lesssim 1$) continues forever in an E-dS universe. The spatial extent of structures within which average $\delta\rho/\rho \sim 1$ continues to grow, while the density deep inside of these structures continues to increase well into the $\delta >> 1$ regime. The growth of overdense regions occurs at the expense of the underdense regions, which, in the linear regime experience negative growth, i.e. $|\delta| \propto a$, but $\delta < 0$.

2.2.3 Growth of DM perturbations in an open, $\Omega_\Lambda = 0$, $\Omega_K \neq 0$ Universe

Friedmann equation takes the form,

$$H^2 = H_0^2 \Big[\Omega_m a^{-3} + (1 - \Omega_m) a^{-2}\Big], \tag{17}$$

Curvature will begin to dominate dynamics at epochs after this condition: $\Omega_m a^{-3} = (1-\Omega_m)a^{-2}$. For $\Omega_m \sim 0.3$ this will happen at $z \sim 1$; for $\Omega_m \sim 0.2$ it will happen at $z \sim 3$. We can assume that during the epoch when curvature dominates, $\Omega_m a^{-3}$ will be negligible compared to $(1-\Omega_m)a^{-2}$, so that $\rho_0 \sim 0$, the matter density contribution to the expansion dynamics is tiny, and can be neglected. The Jeans eq. 13 now becomes,

$$\ddot{\delta} + 2H\dot{\delta} = 0. \tag{18}$$

[1] In the next few sections we will drop subscript k in δ_k.

From the Friedmann eq. 17 we see that when curvature dominates $H \propto a^{-1}$, or $\dot{a}$ =const, therefore $t \propto a$, so that $Ht = 1$. With this, eq. 18 has two solutions,

$$\delta \propto A \times \text{const} + B\, t^{-1} \tag{19}$$

The 'growing mode' solution in this case is the least rapidly decaying one, $\delta \propto$ const, and implies that the amplitude of the density fluctuations in the linear regime is not going to change once curvature comes to dominate the global dynamics of the Universe.

In an open Universe no growth of the density fluctuations in the linear regime took place for a substantial portion of the age of the Universe, since $z \sim 1 - 3$.

2.2.4 Growth of DM perturbations in a flat, $\Omega_\Lambda \neq 0$, $\Omega_K = 0$ Universe

Friedmann equation takes the form,

$$H^2 = H_0^2 \Big[\Omega_m a^{-3} + \Omega_\Lambda \Big] \tag{20}$$

In this model, just like in the open Universe, growth of density perturbations will virtually stop when Λ begins to dominate, i.e. when $\Omega_m a^{-3}/\Omega_\Lambda < 1$. The equation then becomes the same as eq. 18, but the solutions are different because H and t now have a different relation connecting them: from Friedmann eq. 20, $H^2 \approx H_0^2 \Omega_\Lambda$, so Hubble parameter is constant, and the solutions are:

$$\delta \propto A \times \text{const} + B \times e^{-2Ht} \tag{21}$$

When does this happen? For For $\Omega_m = 0.2$ (0.3) and $\Omega_\Lambda = 0.8$ (0.7), $\Omega_m a^{-3}/\Omega_\Lambda < 1$ implies $z \lesssim 0.5$ (0.3). So for the same present day matter density Ω_m, cosmological constant dominated models allow longer period for the growth of structure compared to open (negatively curved) models.

It is important to note that here we are talking about growth shutting off on linear length scales. At the present epoch in our own Universe these are scales larger than about 20-40 Mpc, i.e. scales of superclusters, voids, and larger. The shutting off of growth on these scales does not mean that the dynamical evolution *within* non-linear structures $\delta \gg 1$ that are already assembled has stopped: these objects continue to evolve until they virialize. Also, interactions between objects, like galaxy collisions, etc, can stimulate additional dynamical evolution in mildly non-linear regions.

2.2.5 Growth of sub-horizon DM perturbations during radiation dominated epoch

During radiation domination the largest contribution to the energy density is from photons, which, being relativistic, do not cluster. That means that the last term in eq. 15 is 0, because $\delta_{tot} = \delta_{DM} + \delta_{baryons} + \delta_{photons} \approx \delta_{photons} \approx 0$. The Jeans equation reduces to

$$\ddot{\delta} + 2H\dot{\delta} = \ddot{\delta} + \frac{1}{t}\dot{\delta} = 0 \tag{22}$$

The second step follows because $a \propto t^{-1/2}$ during this epoch, so $H = 1/(2t)$. Eq. 22 has two solutions,

$$\delta \propto A \times \text{const} + B\ \ln t \tag{23}$$

Perturbations in particles not coupled to photons (like CDM) grow at best logarithmically during this epoch, which is not very fast at all. Remember that baryons during this epoch are tightly coupled to the photons and so their density perturbations are well represented by an oscillating solution to eq. 15.

Having considered the rate of growth of structure for various combinations of cosmological parameters and various epochs, we see that only in the matter dominated setting structure grows well. All other settings/conditions/epochs are not very conducive to the growth of density fluctuations. The main conclusion we draw is: *the growth of structure begins and ends with matter domination.*

2.2.6 Gravity-related secondary anisotropies in the CMB

Let's look at eq. 11; ϕ is the peculiar (i.e. due to the perturbations) potential in comoving coordinates. If $\Omega_m = 1$, then δ, we now know, grows as a in the linear regime; $\rho_0 \propto a^{-3}$, being the proper density of the unperturbed background, so the RHS is constant in time: peculiar gravitational potential in comoving coordinates does not evolve in an E-dS model, or in an open or Λ cosmologies during the epochs when matter dominates. (In our Universe that's roughly between $z \sim 10^5$ and $z \sim 1$.) This immediately explains late ISW effect: if $\Omega_m \neq 1$ we expect to see potential wells evolving (getting shallower) on large scales, which leads to a signature in the CMB power spectrum.

What about early ISW effect and Rees-Sciama? These also require that the potential be evolving, $\dot{\phi} \neq 0$, but the reasons are different here. The early ISW is due to photons leaking out of potential wells soon after recombination, and will happen in any type of global cosmology. Rees-Sciama is mostly due to the deepening of potential wells of highly non-linear structures, like galaxies and clusters, and this too is independent of the global cosmology.

In all cases of gravity-induced secondary anisotropies in the CMB power spectrum if $\Delta\phi$ is the change in the potential then the primary CMB $\Delta T/T$ spectrum is altered by

$$\frac{\Delta T}{T} \sim \Delta\phi = \int \dot{\phi}\, dt. \tag{24}$$

2.2.7 Other applications of the Jeans analysis

We already used eq. 15 to explain the acoustic peaks in the CMB power spectrum.

The original Jeans analysis was introduced to treat collapse of stars and planets out of gas clouds, and so completely neglected the Hubble expansion. In fact, Hubble expansion was not known at that time, and it was not Sir James Jeans who extended the formalism to the case of expanding background, relevant to cosmology.

3 Evolution of fluctuations in the linear regime

Here we will discuss how the density fluctuations on a range of scales evolve over the entire lifetime of the Universe, i.e. from inflation, through the epoch of matter-radiation equality, z_{eq}, through the epoch of recombination, z_{rec}, to the present epoch. The only constraint is that the scale in question remain linear, meaning that the rms density fluctuations on that scales are small. At early times the horizon size was small, so that a large range of scales that are relevant in cosmology today, at $z = 0$, were outside the horizon (bigger than the horizon). We already calculated the size of the horizon at the epoch when the microwave background was produced, now we can similarly estimate the comoving size of the horizon at the epoch of matter-radiation equality:

$$R_{hor,eq} \approx (\frac{c}{H_0})(\Omega_m\ h^2\ z_{eq})^{-1/2} \approx 20\ (\Omega_m\ h^2)^{-1} \approx 135 Mpc \tag{25}$$

where we have used an earlier result that $z_{eq} \approx 2.4 \times 10^4 (\Omega_m h^2)$.

When the amplitude of fluctuations is small, they can be broken up into modes of different wavenumbers or wavelengths, k or λ, and evolution of each mode can be considered separately, because there is no energy swapping between modes. The evolution depends on whether the mode is larger than the horizon, or not, and whether the epoch in question is radiation or matter dominated. It also matters whether we are talking about fluctuations in cold dark matter or baryons. We will concentrate on CDM because all evidence points to cold (or coldish) dark matter dominating the matter density in the Universe. We have already considered growth of super-horizon size perturbations (Inflation lecture notes), the next subsections deal with sub-horizon size perturbations.

3.1 Sub-horizon scales

Here we summarize the quantitative results we derived in the last section using the Jeans analysis:
• Radiation dominated:
Because the radiation dominates the energy budget, and photons do not cluster, fluctuations in total matter density (mostly CDM) grow slowly, at best logarithmically with time. During this time baryons are coupled to the photons through Thomson scattering, and so their density is oscillating.
• Matter dominated:
→ Before recombination:
Perturbations in dark matter can now grow as $\delta \propto a \propto t^{2/3}$, while perturbations in baryons are still oscillating, and so do not grow on average.
→ After recombination:
Baryons have now been 'released' from photons, and so quickly settle into the dark matter potential wells. From now on perturbations in both baryons and CDM grow at the same rate, $\delta \propto a \propto t^{2/3}$.
• Curvature or Λ dominated:
The amplitude of density perturbations remains constant; growth has stopped.

3.2 Transfer function

Because of the different rates of growth of perturbations described above, the power spectrum, which is a record of the rms amplitudes of perturbations as a function of wavenumber, changes shape compared to its original primordial Harrison-Zel'dovich spectrum, $P(k) \propto k$. These changes are described by the so-called transfer function which relates the primordial spectrum to the processed, i.e. post-z_{eq} spectrum.

If we consider DM only, there exists a special epoch relevant to the evolution of fluctuations in DM. That epoch is matter-radiation equality (MRE, or just equality), and there will be a special length scale as well, corresponding to the size of the horizon at z_{eq}. Lengthscales less than $R_{hor,eq}$ would have entered the horizon before z_{eq}, and so would have undergone a period of slow (logarithmic) growth, whereas scales larger than $R_{hor,eq}$ would have not experienced such a phase. Hence the processed power spectrum has a (smooth) break at $R_{hor,eq}$, and is no longer scale-free. The original Harrison-Zel'dovich shape $\propto k$ is retained on the longest scales only.

To sum up transfer function in a dark matter dominated universe: long wavelength modes always grow well, because when they make the transition from being super-horizon to sub-horizon the Universe is already matter dominated, and so they never have to go through the growth suppressing period of being sub-horizon during radiation-domination. Unlike smaller length scales. The more time a given scale has to spend in the sub-horizon-radiation-dominated growth purgatory the smaller its amplitude is going to be. So the smallest scales are most severely suppressed.

3.3 Evolution of fluctuations after z_{eq}

When matter begins to dominate the total density fluctuations can begin to grow under gravity at a reasonable rate (this is true for super- and sub-horizon scales). The shape of the matter power spectrum $P(k)$ emerging after z_{eq} has a characteristic "triangular" shape. This shape is retained, more or less for the rest of the evolution until the present, but the amplitude of $P(k)$ grows: all scales grow at the same rate, which is $\delta \propto a \propto t^{2/3}$ so long as matter dominates the Hubble expansion dynamics. If it often said that matter-radiation equality marks the beginning of structure formation. Structure formation comes to an end when matter ceases to dominate, at late epochs in non-Einstein-deSitter models.

A notable thing that happens to the matter power spectrum after equality comes from the epoch of recombination. The oscillations that we see as acoustic disturbances in the CMB temperature maps also get imprinted on the DM distirbutions. Just line in the CMB, the length scale of the "imprint" is the sound crossing horison size at recombination. When looked at in the k space of $P(k)$, we see multiple ripples (harmonics), just like in the CMB. Because galaxy clustering and $P(k)$ can be measured at different redshifts, i.e. $P(k, z)$ an interesting recent application of the baryonic acoustic oscillations is to use the sound crossing horizon as a standard ruler. Measure its angular size at different redshifts, and compute the angular diameter distances as a function of redshift. $D_A(z)$ shape is an indication of the global geometry, and especially, Ω_Λ.

4 Beyond the linear regime

4.1 High density regions and virialized dark matter halos

Our next step is to look at regions with $\delta \gg 1$. As one can imagine the dynamics of these regions can be complicated, so we will make simplifying assumptions. Let's consider an overdense spherically symmetric piece of the Universe, and let the background cosmology be E-dS. Imagine dividing up the region into spherically concentric shells. The density increases inward, so that inner most and densest regions will collapse first, then outer regions, etc. The equation of motion for a particle at radius r is $\ddot{r} = -GM/r^2$, where M is the mass contained interior to r. A general parametric solution to this equation is

$$r = A(1 - \cos\theta) \quad \text{and} \quad t = B(\theta - \sin\theta) \tag{26}$$

Angle θ is sometimes called the evolution angle; it is a surrogate for time; constants A and B are to be determined. When $\theta = \pi$ the particle reaches its maximum distance from the center, r_{max}, therefore $A = r_{max}/2$. This radius is the turnaround, and it occurs at later cosmic times for shells further out. In other words, innermost shells are the first ones to turnaround and start collapsing. A relation between A and B can be established if we plug in r from eq. 26 into the equation of motion, $\ddot{r} = -GM/r^2$ this relation leads to the value of B:

$$A^3 = GMB^2 \quad \longrightarrow \quad B = (\frac{r_{max}^3}{8GM})^{1/2} \tag{27}$$

Let us calculate average density within a given shell of the sphere, $\rho(t) = 3M/4\pi r^3$. Substitute t and θ for M and r:

$$\rho(t) = \frac{3}{4\pi G t^2} \frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3} = \frac{9}{2}\rho_0 \frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3} \tag{28}$$

One can show that for small values of θ, i.e. early on, the density scales as t^{-2}, which is the same as for the average E-dS. At turnaround, $\theta = \pi$, $\rho/\rho_0 = 9\pi^2/16 \approx 5.5$, or $\delta \approx 4.5$ This is an important result. So an initially small perturbation grows, first linearly, then faster. If you follow any one particle its velocity with respect to the center of the bump will steadily decrease compared to the Hubble flow. At some point it stops—that's turnaround—and starts to collapse. The turn-around marks the time when the material at that radius decouples from the Hubble flow; from then onwards that material proceeds with its own dynamical evolution. The turn-around radius—the radius within which the average overdensity is 5.5—increases with time, so collapse of objects proceed from inside out.

What happens to the matter once it has turned around and started to recollapse? The θ-parametric model we have followed so far is going to fail us now: as a shell is beginning to fall in towards the center it encounters the interior shells, so shells will cross each other and the simple theory needs to be modified. We won't deal with that here; let us not pay attention to the ensuing messy process of halo virialization; suppose next time we look at the galaxy all the "dust" has settled, and the galaxy attained a state of (relative) stability: the potential has stopped evolving. Now, what is the density inside the portion of the sphere that is virialized? The virial theorem tells us that for virialized objects $KE = -\frac{1}{2}PE$, where PE and KE are the total potential and kinetic energies of the dark matter halo. At the time of turnaround all the energy is in the form of the potential energy. We can use this to deduce that the radius of the virialized structure is half the radius at turnaround, $2r_{vir} = r_{max}$. Furthermore, roughly, it will take as much time for the structure to collapse as it did to attain turnaround, so the ambient density has decreased between the time of turn-around and the time of virialization. By how much? In E-dS $r \propto t^{2/3}$ and $\rho \propto r^{-3}$, so $\rho \propto t^{-2}$. Time doubled since turnaround, and the radius of the structure has halved, so the density contrast has increased by a factor of $2^2 \times 2^3 = 32$ compared to the density contrast at turnaround, which was 5.5. So an object whose average density is $5.5 \times 32 = 176 \approx 200$ times the ambient density is considered to be virialized. As time goes on larger and larger regions become virialized, ans the boundary enclosing an average overdensity of 200 grows larger.

5 Statistics of discrete objects: Press-Schechter formalism

5.1 Introduction

We know that initially small mass density perturbations grow in time; depending on its fractional overdensity δ we can describe a region as being in the linear stage of development, or collapsed, or something in between. We also discussed the formation of individual collapsed dark matter halos that would host galaxies and clusters. Next, we want to know some statistical properties of the population of collapsed objects, for example, how many massive vs. not-so-massive objects there are, how they are distributed in space, how are they moving with respect to each other, etc. In this section we will address the first of these questions: we would like to know how collapsed objects are distributed in mass, i.e. we will derive their mass function, $n(M)dM$, which is the number of objects per unit volume of space in the mass range between M and $M + dM$. Knowing the theoretical prediction for $n(M)dM$ will help us answer questions like: If I have one Coma[2] cluster in a 100 Mpc region around me how many Virgo[3] clusters should I expect? How many galaxy-satellites should a Milky Way-like galaxy have; in other words, how many dwarf galaxies are there per every massive galaxy? How do these predictions compare to observations?

How do we define a collapsed object? Here, collapsed object means $\delta \sim 200$ dark matter halos that harbor galaxies, groups, or clusters of galaxies, i.e. only those entities that formed directly

[2] a massive nearby cluster
[3] a not as massive nearby cluster

out of the Hubble flow, and via gravitational instability. Stars, planets, giant molecular clouds, etc. are not included in this category because they did not form out of the Hubble flow—they formed inside of galaxies, in regions of space that were dynamically dominated by self-gravity of the parent galaxy, and completely oblivious to the Hubble expansion. The following analysis applies to dark matter halos exclusively. However, in our Universe it appears that dark matter halos of all masses are occupied by some sort of light emitting matter: DM halos with $M \lesssim 10^{12} M_\odot$ are occupied by individual galaxies (in general, one galaxy per halo), $M \sim 10^{13} M_\odot$ DM halos are occupied by groups of galaxies, and $M \gtrsim 10^{14} M_\odot$ DM halos are occupied by clusters of galaxies (again, usually one cluster per halo). So the analysis in effect, applies to galaxies, groups and clusters.

5.2 Mass function of collapsed objects

On average, the distribution of mass is assumed to be homogeneous and isotropic. Let the rms dispersion in mass (or density) in spheres that, *on average*, contain $\bar{M}$, and have radius $R^3 = \bar{M}/[(4/3)\pi\bar{\rho}]$, ($\bar{\rho}$ is the average mass density at that epoch) be

$$\sigma_M = \frac{\langle (M - \bar{M})^2 \rangle}{\bar{M}^2} \propto \bar{M}^{-\alpha}, \tag{29}$$

α is related to the power spectrum index, $\alpha = n/6 + 1/2$, and $P(k) \propto k^n$. Since α is positive rms dispersion in mass decreases with increasing scale. This makes sense because on small scales the Universe is lumpy, but gets smoother on large scales. Since the amplitude of δ grows with time, so does the amplitude of σ_M: $\sigma_M \propto t^{2/3}$ in an Einstein-de Sitter cosmology.

At any given time there will be regions in space that are overdense compared to others; these will accrete surrounding particles to form denser objects, which proceed to collapse after they have reached critical overdensity, $\delta \sim 4.5$. As time goes on more and more particles get caught in collapsed objects. In fact, there will be collapsed objects that become 'ingested' into larger objects that are also collapsed. For example, already formed galaxies become 'particles' incorporated into collapsed clusters of galaxies. In other words, we have a hierarchy of objects. In spite of the fact that an ever increasing amount of mass in the Universe becomes trapped in collapsed objects there are still areas of linear growth in the Universe, but typically you have to go to larger and larger spatial scales to find these. Linear regimes, i.e. where fractional overdensity δ does not exceed ~ 1, often surround objects that are just decoupling from the Hubble flow and are beginning to collapse. So linear formalism approximately applies to spatial scales where collapsed objects are just forming.

Consider many randomly placed volumes of size $V \propto R^3 \propto \bar{M}$. Each one of these is characterized by an overdensity δ, and the volumes are large enough so that typical δ's are not too large. Press and Schechter argued that in such a case δ's are Gaussian distributed:

$$p(\delta, V) = \frac{1}{(2\pi\sigma_M^2)^{1/2}} e^{-\delta^2/2\sigma_M^2} = \frac{1}{(2\pi\sigma_M^2)^{1/2}} e^{-x^2}, \qquad x = \frac{\delta}{\sqrt{2}\sigma_M} \tag{30}$$

At a given time perturbations in the high overdensity tail of the Gaussian are just collapsing or have already collapsed. The fraction of collapsed volumes of all masses is

$$P(\delta, V) = \int_{\delta_c}^{\infty} p(\delta, V) d\delta = \frac{1}{\pi^{1/2}} \int_{x_c}^{\infty} e^{-x^2} dx, \tag{31}$$

The threshold $\delta_c = 1.69$ is the *linearized* fractional overdensity[4], independent of the epoch. Linearized means that it is the overdensity collapsed objects would have had had they not gone

[4] The corresponding non-linear overdensity is 200. This is the actual overdensity.

non-linear. The reason we have to use linearized δ here is because we start with the linear theory and eq. 30 is valid when density fluctuations are small. For objects just collapsing the exponent in eq. 31 becomes:

$$x_c{}^2 = \frac{\delta_c{}^2}{2AM^{-2\alpha}} = \frac{2AM_\star{}^{-2\alpha}}{2AM^{-2\alpha}} = \Big(\frac{M}{M_\star}\Big)^{2\alpha}, \tag{32}$$

where we have replaced δ_c with a new parameter, $M_\star$.

The fraction of mass volumes that are just becoming bound, i.e. are just forming at that epoch is $P(M) - P(M + dM)$, it is the fraction of all objects that have formed by that epoch, minus the fraction of objects that have formed during previous epochs:

$$P(M) - P(M + dM) = \frac{dP}{dM}\,dM = \frac{dP}{dx}\frac{dx}{dM}\,dM = \frac{1}{\pi^{1/2}}\,e^{-x_c^2}\,\alpha\,\frac{M^{\alpha-1}}{M_\star{}^\alpha}\,dM \tag{33}$$

To convert this into the *fraction of mass* contained in these objects per unit volume of space we have to multiply by $\bar{\rho}$, and to get *number density* of objects per unit volume we divide by $\bar{M}$. This gives us the mass function,

$$n(M)\,dM = \frac{dP}{dM}\,dM\,\frac{\bar{\rho}}{M} = \frac{\alpha\bar{\rho}}{\pi^{1/2}}\,\frac{1}{M^2}\Big(\frac{M}{M_\star}\Big)^{\alpha} e^{-x_c^2}\,dM \tag{34}$$

The final form of the mass function is

$$n(M)\,dM = \frac{\alpha\bar{\rho}}{\pi^{1/2}}\,\frac{1}{M_\star{}^2}\Big(\frac{M}{M_\star}\Big)^{\alpha-2} e^{-(M/M_\star)^{2\alpha}}\,dM \tag{35}$$

This is the mass function of objects just reaching virialization at time t. The units on both sides of the "=" sign are: per unit volume. The PS mass function is a power law in mass with an exponential cut-off at masses above $M_\star$. The mass function tells us that massive objects are rearer than less massive objects. $M_\star$ is time dependent because of σ_M. As time progresses larger and larger mass scales reach overdensities necessarily for collapse, and so more massive objects collapse later than less massive objects: galaxies form before clusters, etc.

The derivation of $n(M)\,dM$ made many assumptions, and many relevant effects were not taken into account at all.[5] So it is surprising that the derived mass function agrees very well with the results of high-resolution computer simulations.

6 Clustering of dark matter halos on large scales

In the previous sections we dealt with the mass function of dark matter halos, spanning the mass range from low mass galaxies, to groups and clusters of galaxies. That is, distribution of dark matter halos in mass. We now turn to the distribution of galaxies in space (clustering), and the motion of galaxies on scales $\gtrsim$ few Mpc (dynamics). Both clustering and dynamics can tell us about the global properties of the Universe: Ω's and the shape and amplitude of the matter power spectrum. The hope is that both types of methods give the same results for the parameters, and in fact they do, at least in the general sense. (The details sometimes do not agree; measument errors are a major cause of this.) In other words, we are pretty sure we have the big picture of how structure evolved in the Universe, and all the aspects of the observed large scale structure (LSS) amount to a self-consistent picture.

[5] One of the main weaknesses of Press-Schechter formalism is that it does not treat objects collapsing in underdense regions, i.e. regions with $\delta < 1$. Such regions, voids, do form dark matter halos, but PS ignores them completely. One must correct for this 'by hand'. Since about half of the total mass of the Universe is contained in underdense regions, eq. 35 must be multiplied by a factor of 2. This is a handwaving argument, but one can also show that using more convincing lines of reasoning.

6.1 Correlation functions

Our discussion applies to regions with overdensities δ between ~ 100 to ~ 1. We will treat galaxies as 'particles', because galaxy sizes are small compared to typical distances separating them.

Our first goal is to find a means of quantifying observed clustering. That's usually done using correlation functions. If you are sitting on a galaxy somewhere in our Universe you are more likely to find a galaxy at distance r away from you and within volume dV, compared to a similar situation in a fictitious universe where galaxies are randomly scattered throughout space. In the latter case the probability of finding a galaxy in a volume dV will be independent of distance r away from your parent galaxy. In a universe where galaxy distribution is clustered the probability is a function of r. These statements can be quantified:

$$dP = n\, dV\, [1 + \xi(r)], \tag{36}$$

where dP is the conditional[6] probability, n is the galaxy number density per unit volume, and $\xi(r)$ is the excess probability of finding a galaxy at r away from you, and is called the two-point auto-correlation function. Note that the direction of r is not specified, that's because we have already assumed that our Universe is statistically the same everywhere (at any given epoch). When there is no ambiguity just the 'correlation function' will do; the qualifiers '2 point' and 'auto' remind us that we can also be talking about 3-, 4-, ...n-point correlation functions, and that when two different populations of objects are correlated with one another then cross-correlation functions are used (for example, galaxy-cluster cross-correlation).

One can define $\xi(r)$ somewhat differently: given two randomly placed volume elements, dV_1 and dV_2, the joint probability of finding galaxies in each volume is

$$dP = dV_1\, dV_2\, n^2\, [1 + \xi(r)]. \tag{37}$$

As before, if galaxy distribution is, on average, homogeneous and isotropic then ξ can only be a function of the relative separation of dV_1 and dV_2, not direction; hence ξ is a function of r only. For higher order correlation functions this is not the case; for example, for the 3-point function the shape of the triangle that connects the 3 points is important. Two-point correlation function, being a function of r only is not ideal for capturing information about highly 'non-round' structures. For example, if most galaxies are contained within wall-like or string-like structures then 2-point function will not provide a complete statistical description of mass clustering, and we should also measure 3- and possibly higher order functions (and we will find them to be non-zero). For a Gaussian random process $\xi(r)$ provides complete description of clustering.

$\xi(r)$ was popularized by Peebles in the late 1960's early 70's. Catalogues used back then were not as uniform as what we might expect from today's catalogs, and counting of galaxies was done by eye. Still, reasonable values of $\xi(r)$ were derived, because the projected correlation length happens to be small compared to typical size of a photographic plate. Results found back then are similar to what people find today, using better catalogues and better techniques:

$$\xi(r) \approx (r/r_0)^{-1.8}, \quad r_0 = 5h^{-1}\text{Mpc}, \quad \text{for} \qquad 0.01 \rightarrow 10h^{-1}\text{Mpc}. \tag{38}$$

Here r_0 is the correlation length, i.e. separation (from a galaxy) at which the excess probability of finding (another) galaxy is 1.

If you know the average space number density of galaxies (n galaxies per unit volume of space), then $\xi(r)$ can be used to estimate the total number of galaxies you expect to find within r around

[6] Because you are already assumed to be sitting on a galaxy

any given galaxy:

$$N(<r) = n \int_0^r \; [1+\xi(r)] \; 4\pi \; r^2 \; dr, \tag{39}$$

which is just the integral of eq. 36. One can also find the average density of galaxies within a sphere of radius r centered on a galaxy:

$$\langle N(<r)\rangle = \frac{\int_0^r \; n \; [1+\xi(r)] \; 4\pi \; r^2 \; dr}{\int_0^r \; 4\pi \; r^2 \; dr} = n + \frac{3}{r^3} \; n \; \int_0^r \; \xi(r) \; r^2 \; dr \tag{40}$$

6.2 Correlation function and mass density excess field

The above formalism can be easily extended to clustering of mass. Imagine that continuous mass distribution is broken up into tiny mass elements; eq. 36 and eq. 37 are now understood to apply to mass elements instead of galaxies.

How do we convert the definition in eq. 37 into a practical method of measuring $\xi(r)$ for the mass density field? Eq. 37 suggests that we should measure a product of mass densities at two points separated by r, ρ_1 and ρ_2, and calculate the average of these $\rho_1\rho_2$ over the entire space, i.e. $\langle\rho_1\rho_2\rangle$:

$$dP = \bar{\rho}^2 dV_1 dV_2 (1+\xi) = \langle\rho_1\rho_2\rangle dV_1 dV_2. \tag{41}$$

Here $\bar{\rho} = \bar{\rho}_1 = \bar{\rho}_2$. The two-point correlation function can then be related to the fractional density excess field, δ using the following:

$$\frac{\langle\rho_1\rho_2\rangle}{\bar{\rho}^2} = \langle(1+\delta_1)(1+\delta_2)\rangle = \langle 1\rangle + \langle\delta_1\rangle + \langle\delta_2\rangle + \langle\delta_1\delta_2\rangle = 1 + \langle\delta_1\delta_2\rangle \tag{42}$$

therefore,

$$\xi(r) = \langle\delta_1\delta_2\rangle, \tag{43}$$

because overdensity averaged over all space, $\langle\delta\rangle$ is zero by definition. The separation between δ_1 and δ_2 volumes is r. Correlation function will be non-zero if fractional overdensities δ's are spatially correlated.

6.2.1 $P(k)$ and mass variance

At any given point in space the density contrast has a definite value, $\delta(\vec{x})$, and all the Fourier modes contribute. As we change our location $\vec{x}$ and measure $\delta(\vec{x})$ at every location, what will we find for the rms value of δ?

$$\Big\langle\Big(\frac{\delta\rho}{\rho}\Big)^2\Big\rangle = \langle\delta^2(\vec{x})\rangle \propto \langle\int |\delta_k|^2 \; d^3k\rangle \propto \int P(k) \; k^2 \; dk \tag{44}$$

The above quantity, $\langle\delta^2(\vec{x})\rangle$ is hard to measure in real life because astronomical observations do not give us a continuous 3D field of $\delta(\vec{x})$ values. Instead, we have a distribution of galaxies. A few of the problems are: Some locations $\vec{x}$ will have no galaxies at all; also, the relation between finding a galaxy at a particular $\vec{x}$ and the total mass density at that location is, in general, uncertain. So in reality we have a spotty and noisy representation of $\delta(\vec{x})$. So we have to come up with a measure similar to eq. 44, but where the fluctuations are measured not at a point in space, but in a volume of space, having some size and shape. In that case we will be averaging the density of galaxies over a finite volume, resulting in a continuous, and less noisy $\delta(\vec{x})$ field. The quantity that we measure is called mass variance:

$$\sigma_M^2 \propto \int P(k) \; W^2(kx) \; k^2 \; dk, \tag{45}$$

where $W(kx)$ is the window function. Compare eq. 44 to eq. 45: in the former the window function in real space is a delta function, so the corresponding window function in Fourier space is flat and spans the entire $\vec{k}$ space. A non-delta function window function in real space will then cut out some frequencies from the integral in eq. 45. As a result the value of eq. 45 is always smaller than that of the unfiltered integral over the power spectrum, eq. 44.

There are many choices for $W(kx)$: the simplest is the so-called top hat filter, a sphere. Because of its sharp edges in real space it will have many high frequency modes in its Fourier transform, which is not good. A much better choice is provided by a Gaussian shaped filter; compared to a top hat it gently cuts away the higher modes of $P(k)$. Notice the reciprocal relation between real and Fourier space: applying a Gaussian filter in real space does away with all the space except for the inside of the volume, i.e. leaves a sphere of small $\vec{x}$'s. The corresponding Fourier space picture is: high frequency modes are missing, while the rest remain.

If $P(k)$ is a power law and filter function is Gaussian then $\sigma_M^2 \propto x^{-(n+3)}$, where R is the radius of the Gaussian shaped sphere (same relation as in eq. 29).

6.2.2 $P(k)$ and correlation function

Correlation function and power spectrum are alternative ways of quantifying clustering of mass or galaxies. At any point $\vec{x}$,

$$\delta(\vec{x}) = \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}} \propto \int_{\text{all } \vec{k} \text{ space}} \delta_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}\, d^3k \tag{46}$$

the fractional density excess is a linear superposition of sine and cosine waves of various wavelengths and amplitudes specified by $\delta_{\vec{k}}$. The real space and the Fourier space are both 3D and extend to infinity in all 3 dimensions, and $\vec{k}$ and $\vec{x}$ are real.

From eq. 43 we know that

$$\xi(|\vec{x}|) = \langle \delta(\vec{x}_1)\delta(\vec{x}_1 + \vec{x}) \rangle_{\vec{x}_1}, \tag{47}$$

and the triangular brackets denote average over all space, i.e. all values of $\vec{x}_1$. The value of the 'separation parameter' $|\vec{x}|$, stays constant while the spatial average on the RHS is taken. Since space is statistically isotropic and homogeneous, ξ can be considered to be a function of just x. Instead of δ's we can substitute eq. 46, and simplify the expression,

$$\xi(x) = \langle \delta(\vec{x}_1)\, \delta^\dagger(\vec{x}_1 + \vec{x}) \rangle = \Big\langle (\sum_k \delta_k\, e^{-i\vec{k}\cdot\vec{x}_1})\, (\sum_{k'} {\delta_{k'}}^\dagger\, e^{i\vec{k}'\cdot(\vec{x}_1 + \vec{x})}) \Big\rangle_{\vec{x}_1} \tag{48}$$

Here we have taken advantage of the fact that the correlation function is real; to guarantee a real value on the LHS we multiply together δ and its complex conjugate. The above gives rise to two types of terms, where $k = k'$, and where $k \neq k'$:

$$\xi(x) = \Big\langle (\sum_k |\delta_k|^2\, e^{i\vec{k}\cdot\vec{x}}) \Big\rangle_{\vec{x}_1} + \Big\langle (\sum_{k \neq k'} \delta_k\, \delta_{k'}^\dagger\, e^{-i(\vec{k} - k')\cdot\vec{x}_1}\, e^{i\vec{k}\cdot\vec{x}}) \Big\rangle_{\vec{x}_1} \tag{49}$$

The second term averages out to zero, while the first one does not and can be rewritten as:

$$\xi(x) = \sum_k |\delta_k|^2\, e^{i\vec{k}\cdot\vec{x}} = \int d^3k\, |\delta_k|^2 e^{i\vec{k}\cdot\vec{x}} = \int d^3k\, P(k) e^{i\vec{k}\cdot\vec{x}}. \tag{50}$$

The first and last expressions above state that the correlation function is a Fourier transform of the power spectrum, i.e. the two contain the same information (at least in an ideal case with no observational limitations). The 'inverse' statement is also true,

$$P(k) \propto \int d^3x\, \xi(x)\, e^{-i\vec{k}\cdot\vec{x}}. \tag{51}$$

Fourier k-space is assumed to be isotropic, so we can further simplify eq. 50. For a given $|\vec{k}|$ the volume element of k-space is $d^3k = k^2 \sin\theta \, d\theta \, d\phi \, dk$, and

$$\int_{\text{all } \vec{k} \text{ space}} d^3k \; P(k) e^{i\vec{k}\cdot\vec{x}} = \int_0^\infty dk \; k^2 P(k) \int_0^{2\pi} d\phi \int_0^\pi d\theta [\cos(\vec{k}\cdot\vec{x}) + i\sin(\vec{k}\cdot\vec{x})] \sin\theta. \tag{52}$$

The two parts of the above integral are:

$$\int_0^{2\pi} d\phi \int_0^\pi \cos[|\vec{k}||\vec{x}|\cos\theta] \sin\theta \; d\theta = -\frac{2\pi \sin[|\vec{k}||\vec{x}|\cos\theta]}{|\vec{k}||\vec{x}|}\Bigg|_0^\pi = \frac{4\pi \sin kx}{kx}; \tag{53}$$

$$\int_0^{2\pi} d\phi \int_0^\pi i\sin[|\vec{k}||\vec{x}|\cos\theta] \sin\theta \; d\theta = \frac{2\pi i \cos[|\vec{k}||\vec{x}|\cos\theta]}{|\vec{k}||\vec{x}|}\Bigg|_0^\pi = 0. \tag{54}$$

Therefore,

$$\xi(x) = 4\pi \int_0^\infty dk \; k^2 \; P(k) \; \frac{\sin kx}{kx} \tag{55}$$

The factor $\frac{\sin kx}{kx}$ acts as a 'window function', allowing some modes of $P(k)$ to contribute more to $\xi(x)$ than others. The presence of this window function does not have much effect on the modes with $k^{-1} \gtrsim x$ (large scale modes), so these modes contribute just as is specified by $P(k)$. But, the contribution from the small scale modes, $k^{-1} \lesssim x$, is suppressed by the $(kx)^{-1}$ envelope.

Note that at zero lag, i.e. $x = 0$, $\xi(0) = \langle\delta^2\rangle = 4\pi \int_0^\infty k^2 \; dk \; P(k)$, and we are back to eq. 44.

7 Cosmology with galaxy clusters

Clusters are very special creatures. They are the largest virialized structures in the Universe, and as such can be used to address global cosmological questions.

7.1 Baryon fraction

Most of the baryons in clusters are in the form of $T \sim 10^8$K X-ray emitting gas. Member galaxies, of course, also have baryons, but these are not the dominant component, by a factor of a few. Assuming that on cluster scales (i.e. large scales!) baryons could not have been segregated from other matter (like dark matter) because gravity cannot do that, we are led to conclude that the fraction of baryons in clusters is the same as the average fraction of baryons in the Universe as a whole. The fraction of baryons in cluster is found to be $f_B \sim 15 - 20\%$. Primordial nucleosynthesis, as you will remember, predicts $\Omega_B h^2 \approx 0.02$. These two fractions do not agree if $\Omega_m = 1$. To reconcile baryon fraction in clusters with the global baryon fraction, Ω_m has to be $\Omega_B/fB \approx 0.1h^{-2}$, and for $h = 0.65$, $\Omega_m \approx 0.25$. This is another way to estimate Ω_m.

7.2 Mass determination, and M/L

Today, there are 3 methods to determine the mass of a galaxy cluster: dynamics, X-rays, and gravitational lensing. Lensing is the most accurate method: GL is sensitive to all types of mass, and requires no assumptions to be made about the state of the gas (is it in hydrostatic equilibrium or not?), or the dynamical state of the member galaxies (relaxed or not?). If a cluster is not relaxed, has undergone a recent merger or a close encounter with another cluster, then lensing is your best bet. If a cluster looks relatively isolated, symmetric, and generally undisturbed, then hydrostatic equilibrium of its hot X-ray emitting gas is more or less guaranteed. In that case the temperature of the gas is a good measure of the depth of the cluster's potential well. In fact, in clusters that seem to be undisturbed lensing and X-ray masses agree to within $\sim 20\%$. Dynamical masses (i.e. those derived using velocity information of member galaxies, using the virial theorem) are more problematic. Even if a cluster is relaxed a simple measurement of its line-of-sight velocity dispersion does not tell you the shapes of galaxy orbits, and the derived masses will depend on whether, for example, the orbits are radial with respect to the cluster center, or circular. Obtaining radial velocities of many galaxies over a range of cluster-centric radii helps, but not enough to take care of the problem.

Suppose you have measured masses of a set of clusters. Next you measure their total luminosities (in some waveband) to get M/L. These are typically 300. To get from here to Ω_{matter} we assume that the average M/L of the Universe is same as that of clusters, the logic being that clusters are representative pieces of the Universe, at least as far as M/L is concerned. This is sometimes called the Oort's method. To have $\Omega_{matter} = 1$ one would need a global $M/L \sim 1500$, so we seem to live in an $\Omega_m \approx \frac{300}{1500} \approx 0.2$ Universe.

7.3 Mass vs. light: biasing

Galaxies are known to cluster, i.e. their distribution in space is highly non-random. Since luminous matter is clustered we have reason to believe that all mass (dark matter + luminous) is also clustered. The correspondence between luminous matter and DM is probably not perfect. The simplest assumption is that at any given point in space the fractional overdensity in the space density of galaxies is b times the fractional overdensity in mass (at the same location), $b\,\delta_{mass} = \delta_{gal}$. If b is constant, then rms dispersions in fractional mass and galaxy overdensities are related similarly: $b\,\sigma_{mass} = \sigma_{gal}$. However, in reality the relation between the two is probably more complex; biasing can be non-linear, stochastic, and scale-dependent. What cosmology really wants to know are the clustering properties of mass. Unfortunately, clustering of mass is not easily observable, while clustering of galaxies is. We quantifyed observed galaxy clustering in § 6.

7.4 Space number density of massive clusters

From the derivation of Press-Schechter mass function we know that the space density of the most massive objects, those that live on the exponential part $n(M)dM$ is closely related to the rms dispersion in mass on the corresponding scale; see, for example, eq. 35. σ_M is scale dependent. Consider galaxy clusters whose virial radius is $r_{vir} = 1.5 - 2$Mpc. The corresponding *linearized* size of these clusters will be larger by a factor, $200/(1 + 1.69)^{1/3} = 4.2$, or $r_{lin} \approx 8$Mpc. In other words, we have 'expanded' the clusters to their precollapse 'linear' size; the typical overdensity within these is $\delta = 1.69$. Because the linealized size of these clusters is ~ 8Mpc, σ_M on 8 Mpc scales can be determined from the space density—abundance—of rich galaxy clusters.

It so happens that the r.m.s. dispersion in the number density of *galaxies* in spheres of radius ~ 8 Mpc is $\sigma_{gal} \approx 1$. This was first observationally determined by Davis and Peebles from the CfA redshift survey (in the 1980's?). Because of this coincidence of scales, $\sigma_M = \sigma_{mass}$ on $8h^{-1}$ Mpc scales has a special significance; it is called σ_8, and is often quoted as the normalization of the mass fluctuation power spectrum, $P(k)$.

So, from cluster abundance we can determine the r.m.s. dispersion in *mass* on $8h^{-1}$ Mpc scales; and σ_{gal}, measured to be about 1, is the r.m.s. dispersion in *light* on the same scales. If σ_{gal} and σ_8 are not the same then the light (i.e. galaxies) are a biased tracer of the total underlying mass, and the ratio of the two is called the bias, $b = \sigma_{gal}/\sigma_8$.

If $\Omega_m = 1$ then an estimate of cluster abundance leads directly to σ_8. For other values of Ω_m

the growth rate of fluctuations in the linear regime behaves as $\dot{\delta}/\delta = H\Omega_m^{0.6}$, which implies that a measurement of cluster abundance tells us $\sigma_8\Omega_m^{0.6} = \sigma_{gal}\Omega_m^{0.6}/b$.

The fact that cluster abundance constrains a product of Ω_m and σ_8 can be understood as follows. A certain space number density of massive clusters can form either in a low Ω_{matter} Universe where density fluctuations σ_{mass} are large, or in a high density Universe where density fluctuations are low in amplitude. In either case the amount of mass enclosed ic clusters will be same. Current observational estimates of $\sigma_8\Omega_m^{0.6}$ are around 0.5

The degeneracy between σ_8 and Ω_m can be broken by assuming that $\Omega_m = 1$ (not a good assumption in our Universe), or by studying the evolution in time of the number density of rich clusters. This is because the evolution of number density of objects in redshift is very Ω_m-dependent: high matter density Universe ($\Omega \approx 1$) grows structure at a high rate, so that we expect to see a lot less clusters in the recent past. If we live in a low-Ω_m Universe then the number density of clusters should not decrease much as we look to higher redshifts. Observations strongly favor low Ω_m, around 0.3, and hence $\sigma_8 \sim 0.9$.

8 Bulk flows: mass fluctuations through dynamical means

We know that the amplitude of mass density fluctuations in the Universe grows with time, i.e. overdense regions get more overdense and underdense regions get even more so, compared to the average (smoothed out) density of the Universe at that epoch. Density fluctuations mean that there are gradients in the Newtonian potential, $4\pi G\rho_0 a^2\delta = \nabla^2\phi$, and we know that $\vec{\nabla}\phi$ induces acceleration, which over time will result in peculiar velocities of particles of matter. In this section we will be dealing with particle motions on scales that correspond to linear regime in the density fluctuations, i.e. $\delta \lesssim 1$. Here, our result from Jeans analysis applies: we derived earlier that the perturbed linearized continuity equation reads,

$$\vec{\nabla}\cdot\vec{v} = -a\frac{\partial\delta}{\partial t} = -a\delta\frac{\dot{D}}{D} \tag{56}$$

The last step was accomplished assuming that the time and space dependencies of δ can be decoupled: $\delta(\vec{x},t) = A(\vec{x})D(t)$. In Einstein-de Sitter, $\delta \propto a(t)$, so that $D \propto a$, and therefore $\dot{D}/D = \dot{a}/a = H$. In a more general case where $\Omega_m \neq 1$ the time dependence of D is different; it can be shown that $\dot{D}/D = H\Omega_m^{0.6}$ instead, so that in general

$$\vec{\nabla}\cdot\vec{v} = -aH\Omega_m^{0.6}\delta \tag{57}$$

However, δ is not directly observable because most of the contribution to it comes from dark matter, which is assumed to be traced by light-emitting matter (galaxies) only to within a biasing factor b. What we can determine are the fluctuations in the distribution of galaxies δ_{gal}. In the simplest case of constant biasing the two are related by $\delta_{gal}/\delta = b$. In the linear regime constant b is probably a good approximation. In that case eq. 57 can be rewritten as

$$\vec{\nabla}\cdot\vec{v} = -aH\ (\frac{\Omega_m^{0.6}}{b})\ \delta_{gal}, \tag{58}$$

or, at the present epoch,

$$\vec{\nabla}\cdot\vec{v} = -H_0\ (\frac{\Omega_m^{0.6}}{b})\ \delta_{gal}. \tag{59}$$

On the RHS only δ_{gal} is a function of $\vec{x}$, so we can integrate the above equation to yield,

$$\vec{v}(\vec{x}) = a\frac{H\Omega_m^{0.6}}{4\pi\ b}\int\frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3}\ \delta_{gal}(\vec{x}')\ d^3x' \tag{60}$$

Equations 57 and 60 relate peculiar velocity to the mass density fluctuation field, through Ω_m, the present day matter density. If both $\vec{v}$ and δ are known then Ω_m can be determined.

If we can observationally determine peculiar velocity field, $\vec{v}$ and galaxy distribution δ_{gal} in a volume of radius of, say, $100h^{-1}$ Mpc around us, then we can get $\Omega_m^{0.6}/b$. There are two problems with getting $\vec{v}$: (1) Doppler shifts of spectral lines of galaxies give us *radial* velocities of galaxies; (2) these radial velocities are a combination of the Hubble flow and peculiar radial velocity, $v_r + Hr = z$, redshift z is what we observe. Problem (2) is solved by obtaining distances r to galaxies via methods independent of redshift. Problem (1) means that we need a way of obtaining the other two spatial components of velocity, v_θ, and v_ϕ, given v_r. To get $\vec{v}$ we use the fact that on large spatial scales the peculiar velocity flow is expected to be irrotational, $\vec{\nabla} \times \vec{v} = 0$, so that the peculiar velocity field is, in effect, a gradient of a scalar field, which we will call Φ_v (not to be confused with the gravitational potential):

$$\Phi_v(\theta, \phi, r) = -\int_0^r v_r(r')dr' \tag{61}$$

Having determined the velocity potential at every point in space, each labeled by its coordinate position in spherical coordinates, θ, ϕ, r, we can now evaluate the two plane-of-the-sky components of the peculiar velocity as the gradients of Φ_v:

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi_v}{\partial \theta} \quad \text{and} \quad v_\phi = -\frac{1}{r\sin\theta}\frac{\partial \Phi_v}{\partial \phi} \tag{62}$$

There are a number ways of using eq. 59 or its integral, eq. 60 to get $\Omega_m^{0.6}/b$. For example, if $\delta_{gal}(\vec{x}')$ has been determined from observations, the integral in eq. 60 can be performed. The resulting values are then the *predicted* values of the peculiar velocities given the density field, to within a constant. These predicted values are then compared to the observed values of $\vec{v}$, and the constant of proportionality is thus derived. This constant contains $\Omega_m^{0.6}/b$, which is what we were after.

The whole sub-industry of cosmology that uses eq. 59 and its integral eq. 60 to place limits on Ω_m and b is commonly referred to as 'bulk flows' or 'cosmic flows'.