Contents

Part I
Cosmic Microwave Background

1 Overall properties of CMB

The vital stats are:
Temperature, $T_{CMB} = 2.728$K (Fixsen et al. 1996, Bennett et al. 1996)
Wavelength of the peak: 0.1 cm
Number density of photons at the present time: 412 per cc
Energy density at the present time: 0.25 eV per cc
Average energy per photon at the present time: 6×10^{-4} eV
Baryon-to-photon ratio, by number: $\eta = 2.73 \times 10^{-8}\Omega_B h^2$

At some point in the history of our Universe a time was reached when it became energetically favorable for electrons and protons to be combined into neutral Hydrogen atoms. This is called the epoch of recombination because that's when p^+ and e^- combined (for the first time). It is also called the epoch of decoupling because that's when the radiation decoupled from matter, i.e. scatterings between electrons and photons ceased.

The radiation is thermalized before recombination, by multiple scatterings with the electrons. Before recombination observers won't be able to see anything around them because the mean free path for photons was very short. After recombination photons can fly free, mostly unimpeded by scatterings with electrons. If you were an observer at that epoch, it would be similar to having the fog clear up around you.

An important property of the CMB is that it preserves its black body, Planck shape well after the thermalizing (optically thick) epoch is over. The density of photons in space is $N \sim \nu^2 d\nu$, where $\nu^2 d\nu$ accounts for the integration over the momentum part of phase-space, where the density is n. N drops as a^3 with expansion; each factor of ν decreases as a (because λ increases as a), so n must stay constant; it is an invariantm and hence $\nu \propto T$; for example, the peak of the Planck function increases in wavelength proportionately to $1/T$. The full form of the Planck function, $u(\nu)$, which is proportional to ν^3/n stays the same with expansion; only the black body temperature decreases.

The temperature corresponding to the redshifted radiation scales as $T \propto a^{-1}$, during all epochs (matter or radiation dominated). For example, at the present epoch we see a Planck spectrum of CMB at $T = 2.728K$, but these photons have been free streaming to us, without much hindrance from electrons or other particles since $z \sim 1100$, when the temperature was $T \approx 3000$K.

The existence of the CMB was predicted about 15-20 years before its discovery. First measurements of its radiation intensity function showed that it was a black body, but not necessarily exactly a black body. So there were speculations that CMB has an origin different from what we, today, believe it to be. Perhaps, CMB radiation was starlight, emitted early on, say, $z \sim 100$ and thermalized through frequent interactions with dust. However, it would be very hard to reproduce the single temperature Planck function of CMB by thermalizing with dust particles. As the measurements of the shape of the CMB got better, it got progressively more difficult to make starlight and dust do the job. The COBE satellite data and now the WMAP data show that the spectral shape is Planckian, within a few parts in 10^5: hot early Universe is the only possible place to make such a perfect thermal spectrum.

The importance of this epoch in the history of the Universe is as follows. Most of what we observe comes to us in the form of photons. Before recombination the Universe was optically thick and so no information can come to us via photons from beyond that epoch. We can only see other relics: baryons, light nuclei, neutrinos, etc. After recombination photons can carry the information to us, and we can see using photons. Because of that the epoch of recombination projected onto the sky is often called the last scattering surface. The probes of the epochs before recombination are the neutrinos and gravitational waves.

The power spectrum of the CMB contains a wealth of information. In fact, there are different kinds of information each relating to the cosmic epoch and physical process that gave rise to it. For example, the spectrum depends on the global cosmology, on the properties of clusters of galaxies at all redshifts, and on the properties of dust in the Milky Way galaxy, just to name a few. The angular scale of each piece of information is also relevant, any one given angular scale can have more than one physical mechanism contributing to it.

2 Epoch of recombination

This is when neutral H forms. A similar transition for Helium took place earlier, prior to recombination of Hydrogen, but we will concentrate on H. Since Hydrogen is the dominant chemical element in the Universe (its abundance is $\sim 92\%$ by number) recombination of Hydrogen is a very important event. When did it happen? The particles that we will need to consider as the Universe goes from being almost completely ionized to being neutral are e, p, and neutral H. The number density of baryons is the sum of the number densities of p and H: $n_B = n_p + n_H$. We have neglected the small contribution to n_B from Helium.

Let us define ionization fraction as the ratio of the number density of electrons to the number density of baryons, $x = n_e/n_B = n_e/(n_p + n_H)$. Because baryons and electrons are non-relativistic at this time their number densities are given by this expressionwhich we obtained earlier (see Early Universe lecture notes):

$$n_i \propto (m_i kT)^{3/2} e^{-m_i c^2/kT}, \tag{1}$$

and T is the ambient temperature of the radiation field, and i stands for electron or proton. We would like to find a relation between the ionization fraction x and cosmic time t, or redshift z; we expect x to decrease with t as neutral Hydrogen atoms form. Since T is a monotonically increasing function of z, we could start by finding a relation between x and T.

Consider the following ratio,

$$\frac{n_e n_p}{n_H n_B} = \frac{n_e}{n_B}\frac{n_e}{(n_B - n_e)} = \frac{x^2}{(1-x)} \tag{2}$$

The LHS can be related to T by using eq. 1. Using the fact that the masses of particles are related by

$$m_e + m_p = m_H + B_H, \qquad B_H = 13.6\text{eV}, \quad \text{or,} \quad T_H = 1.6 \times 10^5\text{K}, \tag{3}$$

where B_H is the binding energy of Hydrogen atom, we can rewrite eq. 2 as

$$\frac{x^2}{(1-x)} \propto \frac{1}{n_B}(m_e kT)^{3/2} e^{-B_H c^2/kT} = \frac{1}{\eta\, n_\gamma}(m_e kT)^{3/2} e^{-B_H c^2/kT}, \tag{4}$$

where $\eta = 2.73 \times 10^{-8}\Omega_B h^2$ is the baryon to photon number ratio, and n_γ is the number density of photons, and $n_B = \eta\, n_\gamma$. This equation gives us a dependence of x on T, and so tells us how ionization fraction drops with decreasing temperature, or increasing time. Recombination is not instantaneous, because x does not drop from 1 to 0 at once, but gradually and continuously. We can define the epoch of recombination to be when x has decreased to, say, 10%, and that happens at $z_{rec} \approx 1100$ and $T_{rec} \approx 3000$K. There is a slight dependence of this redshift on the baryon number density, $\Omega_B h^2$, through the η factor. The dependence of x on $\Omega_B h^2$ is relatively small compared to the strong (exponential) dependence of x on T: a change in $\Omega_B h^2$ can be easily offset by a very tiny change in T. Therefore, the redshift of recombination is nearly independent of global cosmological parameters.

It is important to understand why T_{rec} is so much lower than T_H, the temperature corresponding to the Hydrogen atom binding energy. This is due to the large overabundance of photons compared to baryons, or the small value of η. There are about 10^9 photons to every baryon, and so the Hydrogen ionizing photons do not have to come from the center of the Planck function; there are enough of them in the high energy tail of the distribution to ionize all the Hydrogen atoms. Therefore the radiation temperature can be considerably lower than T_H and there will still be as many $E \sim T_H$ photons in the Wein tail as there are atoms. Note that this is the second time we have encountered the effects of tiny η—the first time was BBN.

3 Temperature fluctuations in the CMB

3.1 Many effects contribute

The photons released at the time of recombination travel to us through many epochs, and so the information they bring to us contains the information about everything that has happened to the photons during and since recombination. The information is encoded in the temperature fluctuations, as well the polarization of the CMB. Our job is to disentangle all the effects. There are three main categories of causes for the temperature fluctuations of the CMB. There are [1] primary, which arise during the epoch of recombination, [2] secondary, which arise in the vast span of time after recombination and before the photons reach the nearby Universe, and [3] tertiary, which are fluctuations arising due to the dust and gas in our own Galaxy. Tertiary sources of fluctuations are also often taken to include individual galaxies in the nearby Universe, $z \lesssim$ few. In principle, the rather prominent dipole in the $\Delta T/T$ distribution across the sky would belong in [3], but the above classification usually applies to quadrupole and higher multipoles.

The fluctuations $\Delta T/T = (T - \langle T \rangle)/\langle T \rangle \sim 10^{-5}$ are small; the map of $\Delta T/T$ as a function of position on the sky can be decomposed into spherical harmonics, monopole, dipole, quadrupole, etc, each with its own amplitude.

3.2 Monopole

Overall the average spectrum is very very close to perfect black-body. Because of that, the temperature that results from fitting the whole sky frequency spectrum of the CMB, is the single best descriptor of the CMB. That average temperature is $T_{CMB} = 2.728$K at the present day. The average temperature in the monopole.

3.3 Dipole

When the monopole (average T) is subtracted from the temperature sky map the residual temperature map has a very distinct look to it: the sky is hot in one direction and cold in the opposite. This is the dipole. If angle θ is measured with respect to the direction to the hottest point on the sky, then the temperatures are distributed as

$$T(\theta) \approx T_{CMB}(1 + \frac{v}{c}\cos\theta), \tag{5}$$

where θ is the angle between our direction of motion and the direction in which temperature is measured. This equation comes from considering relativisticaly Doppler shifted rediation. The spectrum is black body in all directions. The dipole is due to our motion with respect to the rest frame of the CMB, with v as the velocity. The actual amplitude of temperature deviations from average is very small, $\delta T/T \sim 10^{-3}$, corresponding to a small velocity, of order of a few 100 km/s.

The velocity of the Solar System with respect to the CMB rest frame is about 390 ± 30 km/s in the approximate direction $l = 265°$ and $b = 50°$. Accounting for the known Solar System motion with respect to the center of mass of the Local Group we get that the Local Group is moving with respect to the CMB at about 627 ± 22 km/s in about the same direction, or, more precisely, towards $l = 276° \pm 3°$ and $b = 30° \pm 2°$, which is the direction towards Hydra-Centaurus.

Why do we think that the dipole arises due to our motion? Its amplitude is substantially larger than that of the next multipole (quadrupole), which would not be the case if both the dipole and the quadrupole had the same physical origin. Also, if the motion of the Local Group is the answer then we should expect to see this motion with respect to cosmologically distributed matter. In fact, we do it: the direction of our motion with respect to a frame defined by thousands of infrared emitting galaxies detected by the IRAS satellite at 1.2 Jansky within ~ 100 Mpc around us, is about 15° away from the direction we get from the CMB dipole. The magnitudes of the two velocity vectors are comparable too.

3.4 Higher multipoles

Next, subtract the dipole from the sky temperature map, the remainder is a patchwork of hot and cold spots with an rms amplitude of $\delta T/T \sim 10^{-5}$. These are what we are really interested in, because they contain a lot of cosmological information. These temperature fluctuations can be quantified in two different ways: using correlation functions or spherical harmonics.

• Correlation function of temperature fluctuations

$$C(\theta) = \Big\langle \frac{\Delta T}{T}(\vec{\theta}_1) \frac{\Delta T}{T}(\vec{\theta}_2) \Big\rangle \tag{6}$$

where the angle brackets represent an average over all locations $\vec{\theta}_1$ and $\vec{\theta}_2$ on the sky, such that the two locations are separated by θ: $|\vec{\theta}_1 - \vec{\theta}_2| = \theta$. One would then plot $C(\theta)$ vs. θ and call it the CMB power spectrum. However, this is not how people usually chose to quantify temperature fluctuations in the CMB. Usually, spherical harmonics are preferred:

• Spherical harmonics decomposition

$$\frac{\Delta T(\theta, \phi)}{T_0} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}\, Y_{lm}(\theta, \phi) \tag{7}$$

where coefficients a_{lm}'s are related to the multipole moments C_l's by

$$\langle a^{\star}_{l'm'} a_{lm} \rangle = C_l \delta_{ll'} \delta_{mm'}, \quad C_l = \langle |a_{lm}|^2 \rangle = \frac{1}{2l+1} \sum_m |a_{lm}|^2 \tag{8}$$

and angle brackets denote an average over many realizations.

The two types of vertical axis choices are related by

$$C(\theta) = \frac{1}{4\pi} \sum_l (2l+1)\, C_l\, P_l(\cos\theta) \tag{9}$$

where P_l's are the Legendre polynomials.

However, what is usually plotted as the vertical axis of the CMB power spectrum is $l(l+1)C_l$, because if the power spectrum of mass density fluctuations (mass power spectrum) is Harrison-Zeldovich then the CMB spectrum on large angular scales would be flat in this representation. In other words, the Sachs-Wolfe part of the CMB spectrum (Sect 4.5.2) would be flat.

4 Primary temperature anisotropies

These arise at the epoch of recombination. They come in two flavors, separated by a special length scale: anisotropies on super-horizon scales, where primordial spectrum of density fluctuations (mostly dark matter) and their associated gravity is the main determining factor, and sub-horizon scales where gravity and fluid pressure are both important, leading to fluid oscillations which result in the CMB acoustic peaks. The separating scale is the size of the sound crossing horizon at recombination, we will derive its size in Section 4.2, but first a couple of preliminaries.

4.1 Sound speed

Compressional waves propagate as disturbances in pressure and density, hence speed of sound is

$$c_s^2 = \frac{\partial p}{\partial \rho} \tag{10}$$

In the early Universe, radiation and baryonic matter were coupled together through frequent photon-electron scatterings, combined with coupling between positively charged protons and negatively charged electrons. Dark matter, which is most probably collisionless, i.e. does not interact with anything except through gravity, would not participate in the pressure and density disturbances that affect the baryon-photon fluid.

Sound speed in the early Universe is determined by considering baryons and radiation, and so

$$c_s^2 = \frac{(\partial p/\partial T)_r + (\partial p/\partial T)_m}{(\partial \rho/\partial T)_r + (\partial \rho/\partial T)_m}. \tag{11}$$

Non-relativistic matter has no pressure to speak of, so $(\partial p/\partial T)_b = 0$. The other terms can be evaluated because we know that $p = w\rho$, and $\rho = \rho_0 a^{3(1+w)}$, and $T \propto a^{-1}$. The result is:

$$c_s^2 = c^2 \frac{(4/3)\rho_r}{4\rho_r + 3\rho_b} = \frac{c^2}{3(1+3\rho_b/4\rho_r)} = \frac{c^2}{3(1+R)}. \tag{12}$$

We introduce $R = 3\rho_b/4\rho_r$, which is proportional to the ratio of the energy density in non-relativistic matter to that in relativistic matter. Eq. 12 reduces to $c_s^2 = c^2/3$ for pure radiation. In the other extreme, radiation contributes very little to the energy density. Accounting for how T and density ρ scale with redshift, we get

$$c_s^2 = c^2\,\frac{4\rho_r}{9\rho_b} \approx 1.1 \times 10^{-5} c^2 \frac{z}{\Omega_b h^2}, \tag{13}$$

in other words, sound speed decreases considerably with cosmic time.

4.2 Size of the sound horizon at recombination

The comoving size of the sound crossing horizon at recombination is the distance sound could travel from $t = 0$ to recombination:

$$R_{sch} = \int_0^{t_{rec}} \frac{c_s\,dt}{a(t)} \approx \frac{1}{\sqrt{3}} \int_{z_{rec}}^{\infty} \frac{da}{H\ a^2} \tag{14}$$

In most models the Universe was matter dominated for many expansions times prior to recombination, so only the matter term in the Friedmann equation is needed here: $H^2 \approx {H_0}^2\, \Omega_m\, (a_0/a)^3$, or $H^{-1} \approx H_0^{-1} \Omega_m^{-1/2} z^{-3/2}$. Using this in eq. 14 we estimate the comoving size of the sound crossing horizon at z_{rec} to be

$$R_{sch,\,comov} \approx \frac{2}{\sqrt{3}}\,\frac{c}{H_0}\,(\Omega_m\; h^2\; z_{rec})^{-1/2} \approx 100(\Omega_m h^2)^{-1/2}\;\;\mathrm{Mpc} \approx 300\;\mathrm{Mpc}, \tag{15}$$

using $z_{rec} \approx 1100$, $c/H_0 \approx 3000$ Mpc, $\Omega_m \approx 0.3$, and $h \approx 0.7$.

4.3 Fate of photon-baryon fluid perturbations

Consider some fluid in a gravitational potential well. In our case it is the baryon-photon fluid. The baryons, being heavy and non-relativistic, will try to drag the fluid to the bottoms of potential wells, while the radiation, with photons being relativistic, will want to disperse any concentrations. The dynamics of this situation can be discussed with the help of two relevant physical effects, gravity and fluid pressure, and the corresponding time-scales.

First, gravity. For simplicity, imagine a uniform density sphere. A test particle, starting from rest at the surface of such sphere (ρ) will be accelerated towards the center of the sphere, eventually reach its center, go the other side of the sphere, momentarily stop, and reverse direction of motion. Let the whole period of that journey, from the starting point, and back to the starting point be t_0. If the particle were to orbit the sphere, the period of the orbit will also be t_0. So t_0 is a characteristic time scale for a uniform density sphere. A quarter of t_0 is called the dynamical time, the time it would take the test particle to reach the center of the sphere. From Newtonian mechanics dynamical time can be evaluated to be

$$t_{dyn} = \sqrt{\frac{3\pi}{16 G\rho}}, \tag{16}$$

and is a function of ρ only, independent of the radius of the sphere.

Next, fluid pressure. Pressure waves propagate at the sound speed. Sound crossing time is just what the name implies,

$$t_{sct} = \lambda / c_s, \tag{17}$$

where λ is the size of the system.

If the dynamical time is much shorter than sound crossing time then perturbation will succumb to gravity and collapse into a bound object, in a time comparable to t_{dyn}.

If sound crossing time is much shorter than the dynamical time, then a perturbation in the fluid (density enhancement or deficit) will tend to get dissipated by pressure waves. But since gravity never gives up, the battle between the two opposing forcing will go on, resulting in oscillations.

The dividing line between the two types of behavior can be expressed in terms of a scale length, called the Jeans length (and the corresponding Jeans mass). The exact derivation of the Jeans length is given elsewhere, but its approximate size can be deduced by equating $t_{dyn} = t_{sch}$; the resulting λ is λ_J. A more correct derivation would give us,

$$\lambda_J = c_s \sqrt{\frac{\pi}{G\rho}} \tag{18}$$

At recombination, the baryonic Jeans length is

$$\lambda_{J,\,prop} = c_s \sqrt{\frac{\pi}{G\rho}} = c_s \sqrt{\frac{\pi}{G}\Big(\frac{1}{\Omega_b}\cdot\frac{8\pi G}{3{H_0}^2}\cdot\frac{1}{{z_{rec}}^3}\Big)} \approx 3\,\frac{c}{H_0}(\Omega_b\, h^2)^{-1/2}\, {z_{rec}}^{-3/2} \approx 2.4\,\mathrm{Mpc} \tag{19}$$

In terms of comoving length this is

$$\lambda_{J,\,comov} = \lambda_{J,\,prop} \cdot z_{rec} \approx 2500\,\mathrm{Mpc} \tag{20}$$

The Jeans length during recombination is larger than the horizon scale, by a factor of $\Omega_m/\Omega_b)^{1/2}$ (compare to eq. 15), so we would expect to have fluid oscillations on all sub-horizon scales, including the scale that just comes inside the horizon at recombination. The dynamics of the fluid on the horizon scale will translate into the first acoustic peak in the CMB power spectrum.

4.4 Angular size of the sound horizon at recombination

Let us evaluate this for a spatially flat Universe model, $\Omega_m + \Omega_\Lambda = 1$. There are two pieces to the calculation. First, the angular diameter distance to the surface of last scattering, which is just $D_A(z_{rec})$, and z_{rec} was calculated earlier. For an approximate calculation we can assume that Ω_Λ's contribution to H will be small because it is only relevant for a small fraction of the whole range of z's. Setting $\Omega_\Lambda = 0$, we get

$$D_A \approx (1+z_{rec})^{-1}(\Omega_m h^2)^{-1/2}\int_{z_{rec}}^{0}(1+z)^{-3/2}dz \approx 2\,(\Omega_m h^2)^{-1/2}\,z_{rec}^{-1} \tag{21}$$

We have also assumed that z_{rec} to be very large compared to 1.

The angular size seen by us at the present time is

$$\theta_{sch} = \frac{R_{sch}}{(1+z_{rec})\,D_A} \approx (3z_{rec})^{-1/2} \approx 1^\circ \tag{22}$$

This is the size of the first acoustic peak in the CMB power spectrum. This angular size is independent of the values of Ω_m and Ω_Λ so long as they add up to 1, i.e. if we live in a flat Universe. In CMBology angular scales are represented by multipoles, $l \approx \pi/\theta$. So for a flat Universe this is always true: $l \approx 200$. The reverse is also true: if $l \approx 200$, then we live in a flat Universe. From observations, l is, in fact, close to 200, so we live in a flat Universe. Recall that inflation *predicts* a flat Universe. CMB observations *show* that our Universe is flat. This does not mean that inflation took place, but it does support inflation.

Furthermore, it can be shown that if the Universe is open, i.e. negatively curved, with only dark matter (Ω_m), and no cosmological constant, then $l \approx 200\,{\Omega_m}^{-1/2}$. The dependence of l of the first acoustic peak on Ω_m in the flat and open Universe cases makes it a useful diagnostic: it is often said that the location of the first acoustic peak is sensitive to the curvature of the Universe.

4.5 Super-horizon scales: Sachs-Wolfe effect

4.5.1 Relation between $\Delta T/T$ and gravitational potential

On superhorizon scales the bulk motion of the baryon-photon fluid in response to the gravitational potential is not important. The relation between the depth of the potential well, ϕ, and the corresponding fractional temperature change, $\Delta T/T$ determines the shape of the CMB power spectrum at large angular scales, or low l's. A photon escaping from a potential well is subject to two main effects, both of which are due to the relativistic time dilation:

$$\frac{\Delta T}{T}(\vec{r}) = \left.\frac{\Delta T}{T}\right|_{grav\,1} + \left.\frac{\Delta T}{T}\right|_{grav\,2} = \phi(\vec{r}) - \frac{2}{3}\phi(\vec{r}) = \frac{1}{3}\phi(\vec{r}) \tag{23}$$

The first term is gravitational red or blueshifting of photons climbing out of potential wells or descending from potential peaks, respectively. In other words, time interval between wavecrests changes. We saw while discussing GR that $\Delta\nu/\nu = \phi$. Because the Planck shape remains unchanged, as we saw in Section 1, $\Delta\nu/\nu = \Delta T/T$. Thus $\Delta T/T = \phi$ is the first term in eq. 23.

The second term is due to the gravitational time dilation of photons climbing out of potential wells, and the opposite effect for photons descending from potential peaks. In both cases, this term is $\Delta t/t = \phi$. How do we tie this to the temperature fluctuation? Because of non-zero Δt the protons we see today that originally come from potential wells left earlier (at an earlier cosmic time) than the photons that originated in potential peaks. So we see photons from wells as they were during an earlier cosmic epoch, when the average temperature was higher everywhere. Using $\rho \propto a^{-3}$ (appropriate for the matter dominated epoch), and $T \propto a^{-1}$, we see that $\Delta T/T = (-2/3)\Delta t/t = (-2/3)\phi$. This explains the second term in eq. 23.

This is the relation we have been seeking, called the Sachs-Wolfe effect. Note that the two effects contributing to eq. 23 work in opposite directions. The first effect, by itself, would make a gravitational potential well appear as a cold spot, compared to average temperature. The second effect, time dilation, wants to make potential wells appear as warm spots. The first effect dominates over the second, and so the overall result is that gravitational potential wells (negative ϕ) look colder than average, $\Delta T/T < 0$ on these large angular scales. The potential peaks appear as warmer than average spots.

The uniformity of the microwave sky temperatures at separations greater than about a degree, $\frac{\Delta T}{T} \sim 10^{-5}$, is a testament to the uniformity of the expansion scale factor a in different regions of the Universe, within the currently observable volume. This reinforces our assumption of the cosmological principle, the isotropy and homogeneity of our Universe.

4.5.2 Temperature fluctuations as a function of angular separation on the sky

Having derived the value of $\frac{\Delta T}{T}$ for a given $\vec{r}$, we now want to know what that implies for the spectrum of $\frac{\Delta T}{T}$ as a function of angular separation θ on the sky.

When discussing inflation we saw that at a given epoch rms fluctuations in gravitational potential $\phi \propto \sigma_M\, M^{2/3}$. Using same notation as we did then, we have $\sigma_M \propto M^{-\alpha_i}$, and $\alpha_i = n_i/6 + 1/2$. So,

$$\phi\,|_{\rm rms} \propto M^{-\alpha_i+2/3} \propto \lambda^{3(-\alpha_i+2/3)} \propto \lambda^{(1-n_i)/2} \tag{24}$$

The rms fluctuations in ϕ as a function of linear scale can be translated into $\phi\,|_{\rm rms}$ as a function of angular separation on the sky, θ,

$$\phi\,|_{\rm rms} \propto (\theta D_A)^{(1-n_i)/2} \propto \theta^{(1-n_i)/2} \tag{25}$$

where the last step takes advantage of the fact that recombination happened at the same epoch everywhere, and hence at the same D_A, and was relatively short in duration. Combining eq 23 and eq. 25, we derive that

$$\frac{\Delta T}{T}\Big|_{SW}(\vec{r}) \propto \frac{1}{3}\theta^{(1-n_i)/2}. \tag{26}$$

For Harrison-Zeldovich spectrum $n_i = 1$. Now, $\ell(\ell+1)C_\ell$ is proportional to the typical squared temperature fluctuations on that scale, and so the CMB power spectrum at large angular separations (low l) is independent of angle θ.

4.6 Sub-horizon scales: Acoustic peaks

4.6.1 Qualitative description

Prior to recombination

Photons and baryons are tightly coupled together through the electrons, forming a single dynamical entity, through which pressure waves can propagate. Well before recombination the sound velocity is about $c_s^2 = 1/3$. Because the fluctuations in the potential which are due mostly to the dark matter are small, we can decompose the potential field into separate Fourier k-modes and treat each mode separately (assume no mode coupling, no transfer of energy between modes). As time progresses larger and larger scales become smaller than sound crossing horizon. As each scale crosses inside the horizon the fluid in the comparably sized potential wells can be treated as oscillating. The oscillations continue, with gravity trying to compress the fluid and the pressure trying to disperse it. Fluid in each of the many potential wells of a given k-mode starts compressing at the same time (when that mode becomes sub-horizon), i.e. *in phase*, thus oscillations are in phase everywhere, for each k-mode.

At recombination

The oscillations continue as long as the photons (that provide pressure) are tightly coupled to baryons (which respond to gravity). However, there comes a time when the ambient temperature drops to a level where there aren't enough high energy photons to keep hydrogen ionized, and recombination takes place. The oscillations cannot continue any longer, since photons are now free from baryons. CMB photons are free-streaming from then onwards, until we detect them. So CMB photons give us a snapshot of the state of the fluid at recombination. (Baryons are now free to fall into potential wells, with no opposing force due to photon pressure.)

The k-mode that happened to be the right size to be hosting maximum compression at the time of recombination will be seen in this snapshot as a network of hot spots (potential wells) and cold spots (potential peaks), with the spacing equal to the sound crossing horizon. This coherent spatial temperature structure at the last scattering surface will translate into a peak in the temperature power spectrum, the so called first acoustic peak. In other words, the peak in power spectrum exists because we cross-correlate $\Delta T/T$ on the sky and thus see the collective signature of these hot and cold spots. Of course there are other k-modes as well: fluid in the wells that are half the size of the sound crossing horizon will be caught at its maximum rarefaction phase, giving rise to the second acoustic peak. Both compression and rarefaction contribute to the temperature power spectrum because the sign of $\Delta T/T$ does not matter for $(\Delta T/T)^2$, only its amplitude matters. In-between modes will be caught at the in-between phases, with not so extreme temperature deviations.

It is important to distinguish two categories of fluid "behavior":

- compression in potential wells and rarefaction in potential peaks,
- rarefaction in potential wells and compression in potential peaks.

The first results in the odd peaks (first acoustic peak, third, etc) in the CMB temperature power spectrum, while the second results in the even peaks. There is an important difference between these two cases. In the presence of baryons rarefactions in potential wells cannot attain as high an amplitude as compressions because baryons have inertia, and want to stay at the bottoms of potential wells, and away from peaks. So, compressions in potential wells have higher $|\Delta T/T|$ than rarefactions in potential wells. This baryon drag results in the odd peaks of the CMB power spectrum having higher amplitude than the even peaks.

4.6.2 Somewhat quantitative description

For the sub-horizon scales the motion of the fluid, as it sloshed around in the potential wells, cannot be neglected because the fluid can traverse these small scales at least once in the age of the Universe. There are three contributions to the temperature fluctuations at any given spatial location before recombination: gravitational frequency shift, temperature change due to the compression or rarefaction of the fluid, and the doppler term due to the motion of the fluid:

$$\frac{\Delta T}{T}(\vec{r}) = \left.\frac{\Delta T}{T}\right|_{grav} + \left.\frac{\Delta T}{T}\right|_{th} + \left.\frac{\Delta T}{T}\right|_{dop} = \phi + \frac{1}{3}\frac{\delta\rho}{\rho} - v_{\parallel} \tag{27}$$

The gravitational potential does not evolve much during recombination, because the Universe is in the matter dominated phase, with dark matter contributing the most to the energy density content of the Universe. Dark matter is not tied to the baryon-photon fluid through interactions therefore the first term is constant for a given location in space. The other two terms do change with oscillations, and we will consider them in turn.

The second term in eq. 27, $\Delta T/T|_{th} = \frac{1}{3}\delta\rho/\rho$ can be understood as follows. Because the photons are coupled to the baryons the densities of the two scale proportionately, $n_\gamma \propto \rho_b$. For black body radiation number density of photons, $n_\gamma \propto \int n\, dp^3$, where n is given by eq. **??**. $\nu \sim T$, and $dp = d(h\nu/c)$, so $n_\gamma \propto T^3$, and so $\rho_b \propto T^3$. This gives us the equation at the top of this paragraph.

The last term in eq. 27 is the Doppler motion shift; we'll discuss it later.

The equation for the evolution of a single k-mode of the perturbation in density, δ is the result of the Jeans analysis of gravitational instability in the linear regime (linear means that the fractional amplitude of flustuations is small):

$$\ddot{\delta}_k + 2(\dot{a}/a)\dot{\delta}_k + ({c_s}^2k^2 - 4\pi G\rho)\delta_k = 0 \tag{28}$$

In our approximate treatment the Hubble expansion term can be neglected. Using perturbed Poisson equation the term $-4\pi G\rho\delta_k$ can be replaced by $k^2\phi_k$. Eq. 28 becomes,

$$\ddot{\delta}_k + {c_s}^2k^2\delta_k + k^2\phi_k = 0 \tag{29}$$

which is an equation of a driven harmonic oscillator, with a constant driving force provided by the gravitational potential. The solution is of the type

$$\delta_k = A\cos(c_skt) + B \tag{30}$$

B can be determined by obtaining first and second time derivatives of δ_k and plugging them into eq. 28: $B = -\phi_k/{c_s}^2 = -3\phi_k\ (1+R)$. Velocity of sound is given by eq. 12 in terms of R. A in eq. 30 is obtained using initial conditions: from eq. 23 we know that $\Delta T/T$ at small k (large spatial scales) is $\phi_k/3$. For $k \to 0$ eq. 30 becomes $\delta_k = A + B$, and eq. 27 becomes $\phi_k + (1/3)(A+B)$. So in the limit of large scales we have $\phi_k/3 = \phi_k + (1/3)(A+B)$, or $A = \phi_k(1/{c_s}^2 - 2) = \phi_k\ (1+3R)$. So, density fluctuations are

$$\delta_k = \phi_k(1/{c_s}^2 - 2)\ \cos(c_skt) - \phi_k/{c_s}^2 = (1+3R)\phi_k\ \cos(c_skt) - 3(1+R)\phi_k. \tag{31}$$

To get the second half of the above equation we used eq. 12. Omitting the doppler term for now, the temperature fluctuations are given by

$$\left.\frac{\Delta T}{T}\right|_{grav} + \left.\frac{\Delta T}{T}\right|_{th} = \phi_k + \frac{1}{3}\delta_k = \frac{(1+3R)}{3}\phi_k \cdot \cos(c_skt) - R\phi_k \tag{32}$$

Now the doppler term. The velocity of the fluid due to its oscillations can be obtained using continuity equation. The latter relates the rate of change of density to the spatial gradient of the velocity: $d\rho/dt = -\rho\vec{\nabla}\cdot\vec{v}$. This becomes

$$\dot{\delta} = -\vec{\nabla}\vec{v} = -\vec{\nabla}\sum_k \delta_v e^{-i\vec{k}\cdot\vec{r}} = \sum_k\ i\vec{k}\cdot\vec{v}_k \tag{33}$$

The two last steps show that if velocity field can be represented as a Fourier sum over independently evolving k-modes, then taking the spatial gradient is the same as multiplication by $-i\vec{k}$. Because we are dealing with each k-mode separately, eq. 33 allows us to relate δ_k and $\vec{v}_k$. Statistically speaking, all three spatial directions have the same rms value of velocity, so the Doppler line-of-sight contribution due to the motion of the fluid is:

$$\left.\frac{\Delta T}{T}\right|_{doppler} = \frac{|\vec{v}_k|}{\sqrt{3}} = \frac{(1+3R)}{3(1+R)^{1/2}}\phi_k\cdot\sin(c_s kt). \tag{34}$$

As expected the maximum amplitude of this term is $\pi/2$ out of phase with the maximum amplitude of eq. 32.

In the absence of baryons, $R=0$, eq. 32 and eq. 34 reduce to cosine and sine of the same angle, and so when added up in quadrature to produce the power spectrum, yield a flat CMB temperature power spectrum. The flat spectrum results because without baryons the treatment of potential wells and peaks is completely symmetric. Baryons break that symmetry because they definitely prefer the bottoms of potential wells, and so enhance the amplitude of compressions in the wells, and rarefactions at the potential peaks, over the reverse situation.

In eqs. 32 and 34 $(c_s kt)\to 0$ is the Sachs-Wolfe limit. The doppler term is negligible in this limit.

Values of $(c_s kt) = \pi,\ 3\pi,\ 5\pi, ...$ correspond to the 1st, 3rd, 5th,... acoustic peaks. These are the odd peaks which we expect to be extra strong in the presence of baryons:

$$(c_s kt) = \pi: \qquad \left.\frac{\Delta T}{T}\right|_{grav} + \left.\frac{\Delta T}{T}\right|_{th} = -\frac{(1+6R)}{3}\phi_k. \tag{35}$$

So these are enhanced by a factor of $(1+6R)$ compared to a no-baryons case of $R=0$.

Values of $(c_s kt) = 2\pi,\ 4\pi, ...$ and other even peaks are expected to be smaller:

$$(c_s kt) = 2\pi: \qquad \left.\frac{\Delta T}{T}\right|_{grav} + \left.\frac{\Delta T}{T}\right|_{th} = \frac{1}{3}\phi_k \tag{36}$$

The doppler peaks are $\pi/2$ off in phase from the potential peaks, and so to get their maximum amplitude we need:

$$(c_s kt) = \pi/2: \qquad \left.\frac{\Delta T}{T}\right|_{doppler} = \frac{(1+3R)}{[3(1+R)]^{1/2}}\phi_k \tag{37}$$

We see that Doppler term is also enhanced, but not as much as the gravitational potential terms. ALl these differences in the amplitude of various temperature and doppler terms in the presence of baryons, imply that the final CMB power spectrum will not be flat, but will have distinct peaks.

4.6.3 How to do it exactly

Calculating the actual power spectrum of temperature fluctuations of the CMB is, of course, more complicated than our simplified treatment. There are several components in the primordial soup prior to recombination: baryons, photons, electrons, neutrinos and dark matter particles. Each component obeys their own Boltzmann transport equations that describe how the phase space density of each component changes with time in the presence of interactions with other species. Because of interactions all these equation are coupled, and have to be evolved together from an early epoch, through recombination, to get the CMB spectrum. Codes exist to do that, the best known being CMBFAST.

4.7 Very small scales: projection effects, and photon diffusion

Small scales are affected by a number of effects, all of which smear the fluctuations, thereby reducing the amplitude of the CMB power spectrum at large l.

Projection involves getting from the 3D k-mode based formalism to the sky-projected 2D l-mode description of the CMB. The thickness of the recombination layer can be estimated to be about $10(\Omega_m h^2)^{-1/2}$ Mpc. This is somewhat smaller than the sound crossing horizon at recombination. Therefore features of this size are projected onto our sky 'one layer thick'. But for smaller scales several of their wavelengths are projected onto the sky to get the contribution to a given l mode, and the resulting smearing reduces the amplitude of fluctuations we observe. This effect becomes important at $l \gtrsim 1000$, or $\theta \lesssim 0.1^\circ$.

As recombination progresses the coupling between photons and baryons gets less tight, so photons tends to diffuse out. This diffusion (how much distance photons can cover during recombination) is a larger fraction of the small potential wells, so photon diffusion process damps the amplitude of temperature fluctuations on smaller scales more than on larger scales.

Both the projection and the diffusion effects act to suppress $(\Delta T/T)^2$ for progressively larger l's.

5 Secondary temperature anisotropies

The physical effects that imprint inhomogeneities on the CMB around the time of recombination are called primary anisotropies. Secondary anisotropies are due to effects that happen to the photons as they make their way from the surface of last scattering to our Galaxy.

Consider a CMB photon just after recombination. That means it just became free but continued to travel unimpeded. Chances are, nothing will happen to this photon for a very very long time (long mean free path), so if we happen to intercept it, it will carry the information that it received when it was last coupled to baryons. The Universe is not completely empty, of course, so some photons do encounter 'obstacles' on the way, for example (1) they can go through a cluster of galaxies, where they can get inverse-Compton scattered from electrons; (2) go through a gravitational well that changed depth during the photon's passage (3) get completely absorbed by a molecule and raise its temperature a bit, etc. etc. These are examples of secondary anisotropies.

First, let us examine gravitationally-induced secondary anisotropies. In a matter dominated Universe perturbation potential does not evolve in time while the density perturbation is in the linear regime (we will derive this result later). As long as that is the case a photon will 'descend' to the same depth and 'climb out' the same height of the potential well, no change in frequency or in temperature will result. If the potential is changing while a photon is crossing the well, then $\delta T/T$ will change.

There are 3 effects associated with this: early integrated Sachs-Wolfe (ISW), late ISW and Rees-Sciama. Early ISW affects photons as they traverse potential wells soon after recombination when the wells have decaying contribution from photons. Late ISW happens in Λ or negatively curved universes. It is due to the grav potentials on supercluster scales becoming shallower at late times (now) because matter-domination gives way to curvature or Λ-domination. These fluctuations appear on large angular scales because they are physically big, and also nearby. Finally, Rees-Sciama is due to potential wells deepening or dispersing because of non-linear evolution effects on the scales of galaxies. Additional gravity-induced temperature fluctuations are due to lensing: nearby galaxies and clusters deflect the paths of light rays thereby redistributing $(\Delta T/T)^2$ power on small scales, and hence smoothing the peaks at large l.

Non-gravitational secondary effects are dominated by reionization when freed-up electrons can scatter off photons. There are two flavors of reionization, local and global. Local takes place in clusters, where hot X-ray emitting gas is the source of high energy electrons. This is Sunyev-Zeldovich effect; there is thermal and kinetic SZ depending on whether thermal or bulk movement of electrons is considered. Global reionization is due to early generation of stars (and QSOs) pouring large amounts of ionizing photons into the Intra Galactic Medium. This will tend to wash out the primary anisotropies at high l, or suppress power spectrum.

And finally, there are some more effects (tertiary effects): point sources in the sky, like radio galaxies, QSOs. In our own Galaxy there's dust, free-free, synchrotron emission, all of which can interact with the CMB photons and hence leave their own mark on the CMB power spectrum.

6 Polarization of the CMB

Around the epoch of recombination electrons are nonrelativistic: their mass is 0.5 MeV while temperature at the time is somewhat above 1 eV. So interaction between electrons and photons is described by Thomson scattering.

Before recombination the radiation field is optically thick, so any given photon scatters from many electrons, and radiation detected by an observer at that epoch would be unpolarized.

At recombination the optical depth decreases so if electrons can somehow impart net polarization to the escaping photons, the polarization acquired by the photons as the result of their last scattering with electrons can be carried by the photons until we intercept them and detect the polarization signal.

How is polarization produced? Consider a potential well of a certain size comparable to the thickness, Δz of the recombination epoch, and an electron at some radius away from the bottom of the well. Say an electron is moving towards the bottom of the well, as a part of the baryon-photon fluid (and so are its electron-friends). Because of the convergent velocity field of the electrons each electron will see a net radiation flux coming at it from directions perpendicular to its direction of motion, i.e. in a torus, so an electron will see a quadrupole radiation field (in addition to other multipoles as well, but we are not interested in those) coming at it. The important feature of the quadrupole part of the radiation field is that the cold and hot intensities are separated by 90 degrees, which allows the radiation, after scattering from an electron to be linearly polarized.

Convergent velocity field is essential to the production of net polarization, so polarization signature will be largest for k-modes (and corresponding l-modes) that have largest fluid velocities at the time of recombination, i.e. polarization peaks will be $\pi/2$ out of phase with the temperature peaks in the CMB power spectrum. The amplitude of the polarization peaks is reduced compared to the temperature peaks by a factor of 10-100.

7 Cosmological parameters from the CMB

• Inflation: Presence of peaks is a strong supporting evidence for inflation, as opposed to topological defects as the seeds for structure. Furthermore, inflation predicts that the polarization fluctuation spectrum should have peaks in between those of the temperature. This is verified as well.
• Global curvature: $l \sim 200$ for a flat Universe; negative curvature would move the peaks towards larger l's (smaller angular scales), see Section 4.2
• Baryon fraction: R used in our discussion is proportional to $\Omega_b h^2$: R determines relative heights of the first and second acoustic peaks; effect is called baryon drag.
• Cosmological constant: location of the first peak cannot tell us the value of Ω_Λ if the Universe is flat, but late integrated Sachs-Wolfe can. There will be extra power on large scales is Λ is present.
• n_i: the index of the matter power spectrum can be obtained from CMB as well, see Section 4.5.2.

8 Dark Ages

After recombination is over and done with, the Universe settles into a very long period, about which we have no direct observational information. Objects must have started collapsing under the action of gravity due to dark matter fluctuations, which go from minor overdensities to overdensities of order 1 and beyond. The only radiation around was the microwave background. Stars have not formed yet, because galaxies and/or globular clusters have not yet collapse sufficiently. Because there are no stars or quasars yet, and hence no bright sources of radiations, this very long epoch is appropriately called the "Dark Ages". Neutral Hydrogen is probably the best observational probe of the dark ages.

The end of the dark ages is marked by the lighting up of first first stars and supernova at redshifts of 5-15. These produce copious amounts of photoionizing radiation, like UV, which reionizes the Universe. Even at the present epoch a considerable amount of gas in between galaxies and clusters is ionized. We know this because we see only small islands of neutral Hydrogen: these islands show up as Lyman-alpha absorption lines in spectra of very distance QSOs. The absorption of radiation by neutral H is called Gunn-Peterson effect. If all the gas was neutral then no radiation shortward of 1216Å would reach us, resulting in a so-called Gunn-Peterson trough in QSO spectra. Such troughs are not seen, hence IGM must be mostly ionized.