Incorrect de Vaucouleurs surface brightness profile

$$I(R) = I_0\ e^{-(R/a)^{1/4}}$$

Let $x = R/a$.
Luminosity within R, or x is

$$L = \int_0^R I(R) 2\pi R dR = \int_0^x I_0 2\pi a x e^{-x^{1/4}} d(ax) = 2\pi a^2 I_0 \int_0^x x e^{-x^{1/4}} dx$$

Set $y = x^{1/4}$, then $y^4 = x$, and $4y^3 dy = dx$.
Luminosity within y is

$$L = 2\pi a^2 I_0 \int_0^y y^4 e^{-y} 4y^3 dy = 8\pi a^2 I_0 \int_0^y y^7 e^{-y} dy$$

Using the integral,

$$\int x^m\ e^{-x}\ dx = -e^{-x}\Big[\sum_{i=0}^{m} \frac{m!}{(m-i)!}\ x^{(m-i)}\Big],$$

and applying the limits, we get

$$L = 8\pi a^2 I_0 (-e^{-y})\Big[\sum_{i=0}^{7} \frac{7!}{(7-i)!}\ y^{(7-i)}\Big] + 8\pi a^2 I_0 7!$$

The first term in the above is for the upper limit, y; the second term is for the lower limit, $y = 0$.

$$L = 8\pi a^2 I_0 7!\ [1 - e^{-y}(\frac{y^7}{7!} + \frac{y^6}{6!} + \frac{y^5}{5!} + \frac{y^4}{4!} + \frac{y^3}{3!} + \frac{y^2}{2!} + \frac{y^1}{1!} + \frac{y^0}{0!})]$$

$$L = 8\pi a^2 I_0 7!\ [1 - e^{-y}(\frac{y^7}{7!} + \frac{y^6}{6!} + \frac{y^5}{5!} + \frac{y^4}{4!} + \frac{y^3}{3!} + \frac{y^2}{2!} + y + 1)]$$

Total luminosity is obtained when $x \to \infty$, or $y \to \infty$,

$$L_{tot} = 8\pi a^2 I_0 7! = 8\pi a^2 I_0 \times 5040 = 126,669.016\ I_0 a^2$$

The fraction of total luminosity enclosed within y is

$$f = 1 - e^{-y}(\frac{y^7}{7!} + \frac{y^6}{6!} + \frac{y^5}{5!} + \frac{y^4}{4!} + \frac{y^3}{3!} + \frac{y^2}{2!} + y + 1)$$

(a)
The fraction of total luminosity enclosed within $x = 1$, or $y = 1$ is

$$f_{[x=0\to 1]} = 1 - e^{-1}(\frac{1}{7!} + \frac{1}{6!} + \frac{1}{5!} + \frac{1}{4!} + \frac{1}{3!} + \frac{1}{2!} + 1 + 1)$$

Note that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$$

So we can tell that $f_{[x=0\to 1]}$ will be very small. The actual value is

$$f_{[x=0\to 1]} \approx 1.03 \times 10^{-5}$$

(b)
The fraction of total luminosity enclosed within $x = 5$, or $y \approx 1.495$

$$f_{[x=0\to 5]} = 1 - e^{-y}(\frac{y^7}{7!} + \frac{y^6}{6!} + \frac{y^5}{5!} + \frac{y^4}{4!} + \frac{y^3}{3!} + \frac{y^2}{2!} + y + 1) \approx 1.661 \times 10^{-4}$$

The fraction of total luminosity enclosed within $x = 10$, or $y \approx 1.778$

$$f_{[x=0\to 10]} = 1 - e^{-y}(\frac{y^7}{7!} + \frac{y^6}{6!} + \frac{y^5}{5!} + \frac{y^4}{4!} + \frac{y^3}{3!} + \frac{y^2}{2!} + y + 1) \approx 5.193 \times 10^{-4}$$

The fraction of total luminosity enclosed between $x = 5$ and $x = 10$,

$$f_{[x=5\to 10]} = f_{[x=0\to 10]} - f_{[x=0\to 5]} \approx 3.532 \times 10^{-4}$$

Correct de Vaucouleurs surface brightness profile

Starting from eq(6.1) on page 243,

$$I(R) = I(R_e)\ e^{[-b\{(R/R_e)^{1/4}-1\}]} = I(R_e)\ e^b\ e^{[-b(R/R_e)^{1/4}]} = I_0\ e^{[-b(R/R_e)^{1/4}]}$$

This is the same as the incorrect expression in the homework, except for the factor b.
We can substitute,

$$x = (\frac{R}{R_e})b^4, \quad \text{so then} \quad I(R) = I_0 e^{-x^{1/4}}.$$

Now we can use this x in the expression for the fractional light obtained on the previous page.

Note that factor b can be evaluated because radius $R = R_e$ is defined to contain half of the total light. The corresponding $x_e = b^4$. Using $f_{[x=0\to x_e]} = 0.5$ we can numerically evaluate the expression for f derived on the previous page. This gives

$$x_e \approx 3466.3, \qquad \text{and so} \qquad b = (x_e)^{0.25} \approx 7.673.$$

So the correct expression for the de Vaucouleurs surface brightness profile is

$$I(R) = I_0\ e^{[-7.673(R/R_e)^{1/4}]}$$

(a)
The fraction of total luminosity enclosed within R_e, i.e. $R/R_e = 1$, or $x = b^4$, or $y = b$ is

$$f_{[R=0\to R_e]} = 1 - e^{-b}(\frac{b^7}{7!} + \frac{b^6}{6!} + \frac{b^5}{5!} + \frac{b^4}{4!} + \frac{b^3}{3!} + \frac{b^2}{2!} + b + 1) \approx 0.5.$$

(b)
The fraction of total luminosity enclosed within $R/R_e = 5$, or $x = 5b^4$, or $y \approx 1.495b$ is

$$f_{[R=0\to 5R_e]} \approx 0.8849$$

The fraction of total luminosity enclosed within $R/R_e = 10$, or $x = 10b^4$, or $y \approx 1.778b$ is

$$f_{[R=0\to 10R_e]} \approx 0.9616$$

The fraction of total luminosity enclosed between $R/R_e = 5$ and $R/R_e = 10$,

$$f_{[R/R_e=5\to 10]} = f_{[R=0\to 5R_e]} - f_{[R=0\to 10R_e]} \approx 0.07677.$$